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Method for Solving the Massive Thirring Model

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ABSTRACT

The Hamiltonian of the massive Thirring model is explicitly diagonalized by formulating a Bethe ansatz for the eigenstates. A general method for computing the energy spectrum is presented.

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The massive Thirring model is the theory of a self-coupled massive fermion field ψ in two dimensions described by the Lagrangian $\mathcal{L}=i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m_0\bar{\psi}\psi - \frac{1}{2}gj^{\mu}j_{\mu}$, where $j^{\mu} = \frac{1}{2}[\bar{\psi},\gamma^{\mu}\psi]$ is the fermion current. The massless case $m_0 = 0$ is exactly soluble and has been extensively analyzed. More recently, considerable evidence has gathered to support the belief that the general case with non-zero mass is also an exactly soluble theory. Much of this evidence hinges on the equivalence between the massive Thirring model and the quantum sine-Gordon theory.[1] At the classical level, the latter theory is exactly integrable by inverse scattering techniques[2] and is found to possess an infinite number of conservation laws. Exact results for the quantized bound state spectrum[3] and S-matrix[4] attest to a corresponding set of conservation laws in the quantum theory. Moreover, by studying a simpler theory (the nonlinear Schrödinger equation), it has been argued[5] that the existence of an infinite number of conservation laws in a two-dimensional field theory is intimately related to the success of a Bethe ansatz as a means of diagonalizing the Hamiltonian of the theory. These arguments suggest that the Bethe ansatz technique might also provide a solution to the massive Thirring model. A different approach which leads to the same attitude is provided by the work of Luther.[6] He pointed out that the massive Thirring model may be considered as the continuum limit of the anisotropic Heisenberg (XYZ) spin chain, with the fermions being identified with spin waves via a Jordan-Wigner transformation. This led Luther to the remarkable observation that, in the appropriate limit, the bound state spectrum of the XYZ chain[7] was identical to the WKB sine-Gordon doublet spectrum.[3] The treatment of the XYZ Hamiltonian is based on its connection with the transfer matrix of the 8-vertex lattice (Baxter model). The methods of Baxter[8] may be described as a generalization of the Bethe ansatz technique. Thus, the solutions of both the classical sine-Gordon equation and the Baxter model suggest a certain strategy for solving the massive Thirring model. With these motivations we have formulated an exact treatment of the massive Thirring model which provides an explicit diagonalization of the Hamiltonian and a method for computing the energy eigenvalue of any physical state. The techniques we employ are largely inspired by the elegant treatment of the 8-vertex model by Baxter[8] and by Johnson, Krinsky, and McCoy.[7] However, by remaining within the continuum field theory, we achieve a considerable degree of simplicity in comparison with the corresponding lattice methods. The relationship between our formalism and that of the 8-vertex model will be considered elsewhere along with a more detailed discussion of the method presented here.

Choosing a basis in which γ^5 is diagonal, we write the Hamiltonian

$$H = \int dx \left\{ -i \left(\psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2 \right) + m_0 \left(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1 \right) + 2g_0 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right\} . \tag{1}$$

Here we have normal ordered H with respect to an unphysical state (denoted $|0\rangle$) which is annihilated by the field operator, i.e. $\psi_1(x)|0\rangle = \psi_2(x)|0\rangle = 0$. The transition from $|0\rangle$ to the physical vacuum $|\Omega\rangle$ is accomplished by filling the Dirac sea. Consider first the Hamiltonian H_0 for free massive fermions, $g_0 = 0$. This is diagonalized by introducing momentum space operators $a^\dagger_{1k}$ and $a^\dagger_{2k}$ and performing a Bogoliubov rotation which mixes them by an angle θ_k where $\cot 2\theta_k = k/m_0$. Eigenstates of H_0 are formally constructed upon $|0\rangle$ by repeated application of the rotated operators which carry energies $\pm(k^2 + m_0^2)^{1/2}$. In particular, the physical vacuum is the state with all negative energy modes filled. Introducing the rapidity β, where $k = m_0 \sinh\beta$ and identifying each filled mode with a point in the complex β plane, the vacuum state may be visualized as a distribution of points along the line $\beta = \alpha + i\pi$. (Here and elsewhere, α is real.) The density of points along this line is determined by imposing periodic boundary conditions in a box of length L and letting $L \to \infty$.

We will now diagonalize the full Hamiltonian (1) by the ansatz

$$|\Phi(\beta_1,\ldots,\beta_N)> \; = \; \int dx_1 \ldots d_N \chi(x,\beta) \prod_{i=1}^{N} A^\dagger(\beta_i, x_i)|0> \tag{2}$$

where

$$A^\dagger(\beta, x) \; = \; \cos\,\theta(\beta)\,\psi_1^\dagger(x) + \sin\theta\,(\beta)\,\psi_2^\dagger(x) \tag{3}$$

with $\cot 2\theta(\beta) = \sinh\,\beta$, and the wave function χ is given by

$$\chi(x,\beta) \; = \; \exp\left[i \sum_{i=1}^{N} m_0 x_i \sinh\,\beta_i \right] \left\{ \prod_{i<j\leq N} [\, 1 + i\lambda(\beta_i, \beta_j)\varepsilon(x_i - x_j)\,] \right\} \tag{4}$$

where $\varepsilon(x)$ is a step function. By applying the Hamiltonian to the state (2) we find that, with an appropriate choice of the function $\lambda(\beta_i, \beta_j)$, it is an eigenstate,

$$H|\Phi(\beta_1,\ldots,\beta_N)> \; = \; \left[\sum_{i=1}^{N} m_0 \cosh\,\beta_i \right] |\Phi(\beta_1,\ldots,\beta_N)> \quad . \tag{5}$$

To show (5), we proceed as we would in the free fermion case, using integration by parts to apply the x derivative in the kinetic energy operator to the wave function. The derivative of the exponential factor in (4) combines with terms from the mass operator to give the right hand side of (5). The x derivative also produces leftover terms from the x dependence of the factor in curly brackets in eq. (4) which are of the same form as those from the interaction. The leftover terms from the kinetic energy operator completely cancel the interaction terms provided that $\lambda(\beta_1, \beta_2) \equiv \lambda(\beta_1 - \beta_2) = -g_0 \tanh \frac{1}{2}(\beta_1 - \beta_2)$. This corresponds to a two-body phase shift

$$\phi(\beta) = 2\tan^{-1}\lambda(\beta) = 2\tan^{-1}\{\cot\mu \tanh \tfrac{1}{2}\beta\}$$

$$= -i \ln\left\{-\frac{\sinh \frac{1}{2}(\beta - 2i\mu)}{\sinh \frac{1}{2}(\beta + 2i\mu)}\right\} \tag{6}$$

where we have defined a constant $\mu = -\cot^{-1} g_0$.

To study the spectral properties of the model, we must impose periodic boundary conditions (PBC's) on the wave functions (4) by requiring that $\chi(x_i = 0) = \chi(x_i = L)$. This leads to N conditions on the β_i's,

$$\exp\left\{-im_0 L \sinh\beta_i\right\} = \exp\left\{i \sum_{j=1}^{N} \phi(\beta_i - \beta_j)\right\} \qquad i = 1,\ldots,N \quad . \tag{7}$$

A detailed analysis of the PBC's (7), leading to the results described here, will be presented elsewhere. We can choose all β's to lie in the strip $-\pi < \mathrm{Im}\,\beta \leq \pi$. As in the free fermion case, the physical vacuum has all β's along the $i\pi$ line, $\beta = \alpha + i\pi$. The log of (7) reads

$$-m_0 L \sinh\beta_i = \sum_{j=1}^{N} \phi(\beta_i - \beta_j) + 2\pi n_i \quad , \qquad i = 1, \ldots, N \quad . \tag{8}$$

For the vacuum state, the distribution along the $i\pi$ line contains no holes, i.e. $n_{i+1} = n_i + 1$. We will always remain in the neutral charge sector, where excited states are obtained by removing points from the $i\pi$ line and placing them in configurations which satisfy the PBC's. These configurations are referred to as "n-strings,"[7] with n = 1, 2, ... For $L \to \infty$ an n-string is a row of points in the rapidity plane at values $\beta_\ell = \alpha_s + i\ell(\pi - \mu)$ with $\ell = (n-1), (n-3), \ldots, -(n-1)$. To discuss the structure of these excitations, we divide the range of coupling into regions labelled by an integer r, where $r\pi/(r+1) < \mu < (r+1)\pi/(r+2)$. In region r, the n-strings which are allowed by the PBC's satisfy $n \leq r + 2$. Each of the two longest strings, $n = r + 2$ and $n = r + 1$, is required by the PBC's to have at least $(n - 2)$ holes directly above it at $\alpha_s + i\pi$. These are states of an unbound fermion-antifermion pair. Each of the other allowed strings, $n \leq r$, is required by the PBC's to have all of its n holes directly above it. These n-string + n-hole excitations for $n \leq r$, are the fermion-antifermion bound states of the model. It is interesting to note that the limit $\mu \to (r \to \infty)$ corresponds to the weak coupling limit of sine-Gordon theory.[1] In this limit the elementary fermions are represented by n-strings with very large n. The correspondence between similar structures (n-body bound states for large n) and classical solitons has been discussed in the context of the nonlinear Schrödinger equation.[11]

To compute the energy of a physical state, we note that, in the vacuum, the points along the $i\pi$ line approach a continuous distribution $\rho(\alpha)$ as $L \to \infty$, whereupon rapidity sums are replaced by integrals. A linear integral equation for the vacuum state distribution $\rho(\alpha)$ is obtained by subtracting adjacent PBC's (8), which gives

$$m_0 \cosh \alpha = \int_{-\Lambda}^{\Lambda} d\alpha' K(\alpha - \alpha') \rho(\alpha') + 2\pi\rho(\alpha) \quad , \tag{9}$$

where $K(\alpha) = d\phi/d\alpha$. Here we introduce a rapidity cutoff Λ which will be taken to infinity after mass renormalization. For an excited state, the presence and location of holes along the $i\pi$ line is determined by the choice of n_i's in (8). Consider the PBC for point ℓ along the $i\pi$ line. Let α_ℓ and α_ℓ' be the real part of its rapidity value in the vacuum and in an excited state respectively. The difference $(\alpha_\ell' - \alpha_\ell)$ is of order $1/L$, so we define $w_\ell = (\alpha_\ell' - \alpha_\ell)L$. As $L \to \infty$, the w_ℓ's approach a continuous function $w(\alpha)$. By subtracting the vacuum PBC from the excited state PBC, and using eq. (9), we obtain an integral equation for the quantity $F(\alpha) \equiv w(\alpha)\rho(\alpha)$,

$$2\pi F(\alpha) + \int_{-\infty}^{\infty} K(\alpha-\alpha')F(\alpha')d\alpha' = \sum_{\ell=1}^{n} \phi(\alpha+i\pi-\beta_\ell^{(s)}) - \sum_{\ell=1}^{n} \phi(\alpha+i\pi-\beta_\ell^{(h)})$$

$$\equiv \Phi_n^{(s)}(\alpha) - \Phi_n^{(h)}(\alpha) \quad . \tag{10}$$

Here n is the number of points removed from the $i\pi$ line to form the excitation. We have let $\Lambda \to \infty$ since the integral is found to be convergent. For a single n-string, the first sum in (10) is over the β's of the string and the second sum is over the holes which are left in the $i\pi$ line. By the previous discussion, if $n \leq r$ we must have $\beta_1^{(h)} = \beta_2^{(h)} = \ldots = \beta_n^{(h)} = i\pi + \alpha_s$. (Of course, these correspond to n different modes in a finite size box which are infinitesimally spaced as $L \to \infty$.) For $n = r+1$ or $r+2$, the holes are located at $\beta_1^{(h)} = i\pi + \alpha_1$, $\beta_2^{(h)} = i\pi + \alpha_2$, $\beta_3^{(h)} = \ldots = \beta_n^{(h)} = i\pi + \alpha_s$.

The physical energies are computed by noting eq. (5) and subtracting vacuum state from excited state eigenvalues,

$$E_n = \sum_{\ell=1}^{n} m_0 \cosh\beta_\ell^{(s)} - \sum_{\ell=1}^{n} m_0 \cosh\beta_\ell^{(h)} + B_n \quad , \tag{11}$$

where

$$B_n = m_0 \int_{-\Lambda}^{\Lambda} \sinh\alpha \, F(\alpha) d\alpha = m_0 \int_{-\infty}^{\infty} dy \tilde{F}(y) \left\{ \int_{-\Lambda}^{\Lambda} e^{i\alpha y} \sinh\alpha \, d\alpha \right\} \tag{12}$$

and $\tilde{F}(y)$ is the Fourier transform of $F(\alpha)$. Eq. (12) represents a "backflow" of the Dirac sea. Other conserved quantities such as momentum may be calculated in a similar way. The equation (10) is solved by Fourier transformation which gives

$$\tilde{F}(y) = \frac{1}{2\pi} [1 + \tilde{K}(y)]^{-1} \left\{ \tilde{\Phi}_n^{(s)}(y) - \tilde{\Phi}_n^{(h)}(y) \right\} \quad . \tag{13}$$

The transform of the kernel is $\tilde{K}(y) = \sinh[(\pi - 2\mu)y]/\sinh \pi y$. The first sum in eq. (10) simplifies by a property of the phase shift (6), giving

$$\tilde{\Phi}_n^{(s)}(y) = \frac{1}{iy} \{ G_{n-1}(y) + G_{n+1}(y) - n \} e^{i\alpha_s y} \qquad n \leq r$$

$$= \frac{1}{iy} \{ G_{n-1}(y) + H_{n+1}(y) - (n-2) \} e^{i\alpha_s y} \qquad n = r+1, r+2 \tag{14}$$

where

$$G_n(y) = \sinh[n(\pi - \mu)y]/\sinh \pi y \quad , \tag{15}$$

$$H_n(y) = \sinh[n(\pi - \mu)y - 2\pi y]/\sinh \pi y \quad . \tag{16}$$

The hole sum gives

$$\Phi_n^{(h)}(y) = \frac{1}{iy} \left(\frac{\sinh[(\pi - 2\mu)y]}{\sinh \pi y} \right) \left[(n-2)e^{i\alpha_s y} + e^{i\alpha_1 y} + e^{i\alpha_2 y} \right] \quad , \tag{17}$$

where $\alpha_1 = \alpha_2 = \alpha_s$ if $n \leq r$. The y integral in (12) is dominated by its nearby singularities. The poles at $y = \pm i$ give a contribution which exactly cancels the first two terms in eq. (11). The other nearby poles at $y = \pm i\gamma$ where $\gamma = \pi/2\mu$ provide the exact expression for the energy.[9] Other pole residues vanish exponentially as $\Lambda \to \infty$. Defining the physical mass [10]

$$m = m_0 \left\{ \frac{e^{(1-\gamma)\Lambda}}{\pi(\gamma - 1)} \tan \pi\gamma \right\} \quad , \tag{18}$$

we find

$$E_n = m \cosh \gamma\alpha_1 + m \cosh \gamma\alpha_2 \quad , \quad n = r+1, r+2 \quad , \tag{19}$$

$$E_n = 2m \sin\left[\frac{n\pi}{2}(2\gamma - 1)\right] \cosh \gamma\alpha_s \quad , \quad n \leq r \quad . \tag{20}$$

In the rest frame $\alpha_s = 0$, eq. (20) gives the familiar sine-Gordon doublet spectrum of Ref. 3. The constant μ is related to the g of Ref. 1 by $2\mu = \pi(2g + \pi)/(g + \pi)$. By a similar calculation, the momentum is

$$P_n = m \sinh \gamma\alpha_1 + m \sinh \gamma\alpha_2, \quad n = r+1, r+2 \quad , \tag{21}$$

$$P_n = 2m \sin\left[\frac{n\pi}{2}(2\gamma - 1)\right] \sinh \gamma\alpha_s, \quad n \leq r \quad . \tag{22}$$

We have described an exact diagonalization of the massive Thirring model Hamiltonian. The method presents attractive possibilities for further study of the Thirring model as well as other field theories which are proven or conjectured to have an infinite number of conservation laws. The explicit expressions for eigenstates, eqs. (2)-(4), provide a new approach to the study of Green's functions, reducing the question to a difficult but perhaps tractable problem of calculating inner products of Bethe wave functions.

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[9] Our cutoff procedure encounters difficulties for $\mu \leq \pi/3$ which will be discussed elsewhere. Here we always assume $\mu > \pi/3$.

[10] Here, the limit $\Lambda \to \infty$ can be related to the lattice continuum limit discussed by Luther (Ref. 6) where the cutoff factor $e^{-\Lambda}$ is analogous to the elliptic modulus ℓ^2 of the eight-vertex model and XYZ spin chain formalism. The precise connection between ℓ^2 and Λ emerges from a study of the critical limit of the eight-vertex model which will be presented elsewhere.

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