Fermi National Accelerator Laboratory

FERMILAB-Pub-78/51-THY
June 1978

Gauge Invariant Quantum Variables in QCD*

ITZHAK BARS†
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
and
Yale University, J. Willard Gibbs Laboratory,
New Haven, Connecticut 06520

AND

FREDERIC GREEN
Yale University, J. Willard Gibbs Laboratory,
New Haven, Connecticut 06520

ABSTRACT

We establish the gauge invariance property of recently proposed variables that lead to flux tubes in the strong coupling limit of quantum chromodynamics. We present a method for deriving the $SU(N)_1 \otimes SU(N)_2 \otimes SU(N)_3$ algebra satisfied by these variables, which was previously guessed in an axial gauge formalism. We make use of Dirac brackets and apply our method to lightcone variables of similar nature.

---

* Research supported in part by U.S. Department of Energy under contract No. EY-76-C-02-3075.

† A.P. Sloan Foundation Fellow.

Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration

I. INTRODUCTION

Recently a new set of variables[1] has been proposed in order to study the properties of Quantum Chromodynamics (QCD). These variables have proven to be useful in the study of the strong coupling limit of the theory. In particular, a description of string-like quantized electric flux tubes is naturally obtained in this formalism. Furthermore, when the theory is placed on a lattice, a precise map can be established[2] between the new variables and the conventional lattice variables.[3] The new variables are local and correspond to "corner" variables as opposed to the bilocal "link" variables of ref. 3 . In terms of the local corner variables it is easy to establish that the lattice quantum theory is very closely related to continuum QCD[2], a result which was difficult to obtain in terms of the link formulation.

In ref. 1 a consistent set of commutation rules satisfied by the new variables was proposed. They were guessed at by demanding consistency with the canonical commutation rules of the independent dynamical degrees of freedom defined in the axial gauge.[4,5] In this paper we present a method for deriving the desired commutation rules, allowing generalizations to similar systems of variables which may be defined more generally in curvilinear coordinates. The method is applied to lightcone variables of similar nature.

The present approach clarifies further the properties of these variables. In particular, it becomes evident that they are gauge invariant and that their commutation rules are a gauge invariant property of QCD. These commutation rules are related to chiral-like local transformations realized on these variables and they should follow in any quantization approach to QCD.

We begin by parametrizing the 4 hermitian and traceless matrices (gauge potentials) $A_\mu = A_\mu^a \frac{\lambda^a}{2}$ in terms of 4 unitary matrices B_μ which have the same number of independent functions.

$$A_\mu = \frac{i}{g} B_\mu^\dagger \partial_\mu B_\mu \text{ (no sum on } \mu) \quad . \tag{1.1}$$

We demand that the B_μ are SU(N) group elements for each μ .

$$B_\mu^\dagger B_\mu = 1 = B_\mu B_\mu^\dagger \tag{1.2a}$$

$$\det B_\mu = 1 \quad . \tag{1.2b}$$

Before proceeding any further let us note some gauge symmetry properties of these variables. The theory is invariant under the local gauge transformation

$$A_\mu \to A_\mu' = U^\dagger (A_\mu + \frac{i}{g} \partial_\mu) U \tag{1.3}$$

where $U(x_\mu) \in SU(N)$. The B_μ's as defined in eq. (1.1) have a much simpler behaviour under gauge transformations:

$$B_\mu \to B_\mu' = B_\mu U \quad . \tag{1.4}$$

Note that the gauge transformation acts on the right of the matrix B_μ. In addition, each B_μ may be transformed from the left by an independent gauge transformation T_μ , such that

T_μ is independent of x_μ for given μ(e.g., T_0 is independent of x_0). That is, for

$$\partial_\mu T_\mu = 0, \quad \text{(no sum on } \mu) \tag{1.5}$$

A_μ remains invariant under the transformations

$$B_\mu \to B_\mu^{'} = T_\mu B_\mu \quad . \tag{1.6}$$

From eq. (1.1), it is apparent that $B_\mu{}^\dagger$, for given μ, is closely related to the gauge transformation that leads to $A_\mu = 0$. In other words, if we take in eq. (1.4) $U = B_0{}^\dagger$ (for example), then $B_0^{'} = B_0 B_0{}^\dagger = 1$ and $B_I^{'} = B_I B_0{}^\dagger$, $I = 1,2,3$ so that $A_0^{'} = 0$ and $A_I^{'} = \frac{i}{g}(B_0 B_I{}^\dagger)\partial_I(B_I B_0{}^\dagger)$. This form makes it evident that the left-handed symmetries T_μ (in this example $T_0(\vec{x})$) are closely related to the remaining gauge invariance after the choice of some linear gauge such as $A_\mu = 0$ for given μ.

We note that by multiplying eq. (1.1) from the right by B_μ for each μ, one obtains a differential equation from which B_μ may be solved in terms of A_μ in the form of a path ordered integral. The symmetry T_μ is related to the fact that the boundary value of the integral may be chosen arbitrarily. In this paper we will take the view that the B_μ are the fundamental local variables and that the A_μ are obtained from them via eq. (1.1) if desired. The consistency of such a point of view will become self-evident as we proceed.

If we are working in a space with a metric $g_{\mu\nu}$, then $A^{\mu} = g^{\mu\nu}A_{\nu}$, and we will define B^{μ} through the relation

$$A^{\mu} = \frac{i}{g} B^{\mu}\partial^{\mu}B^{\mu} \quad . \tag{1.7}$$

Clearly, B_{μ} is <u>not</u> a vector in this space, and generally $B^{\mu} \neq g^{\mu\nu}B_{\nu}$. In this paper we will mainly be concerned with the ordinary Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We will also consider the non-diagonal lightcone metric towards the end of the paper.

We intend to utilize the formalism with a more general metric by introducing a general change of coordinates

$$x^{\mu} = x^{\mu}(u) \quad , \tag{1.8}$$

and work in the u-basis while treating $x^{\mu}(u)$ as a field on the same footing with the other fields in the theory. In this approach $x^{\mu}(u)$ play the role of collective coordinates which obey constraints given through the canonical formalism. One may be able to take advantage of such a formalism in order to work in a string-like basis, since QCD leads to such structures naturally in the strong coupling limit. Work along these lines will be reported elsewhere.

In Section II, we discuss the form of the Lagrangian in terms of the new variables and identify the canonical variables and constraints. In Section III we quantize the theory, solve the constraints and prove that our variables are gauge invariant.

In Section IV we apply our method to lightcone variables of similar nature and derive the lightcone Hamiltonian and commutation rules.

II. LAGRANGIAN AND CONSTRAINTS

We start with the familiar Lagrangian for QCD without fermions:

$$L = -\tfrac{1}{2}\,\mathrm{Tr}\; F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,g[A_\mu, A_\nu] \quad . \tag{2.2}$$

In which we make the substitution (1.1) for A .

As discussed in the previous section, $B_\mu{}^\dagger$(for a given μ) is closely related to the gauge transformation that leads to $A_\mu = 0$. This fact can be utilized to obtain a simple expression for $F_{\mu\nu}$, as follows. Write $F_{\mu\nu}$ in terms of a gauge transformed $F'_{\mu\nu}$, where $F'_{\mu\nu}$ is defined in the gauge $A_\nu = 0$ (i.e., $F'_{\mu\nu}=-\partial_\nu A'_\mu$ for given μ and ν). Then by the above argument,

$$F_{\mu\nu} = B_\nu^\dagger\, F'_{\mu\nu}\, B_\nu \quad . \tag{2.3}$$

Since

$$A'_\mu = B_\nu(A_\mu + \frac{i}{g}\,\partial_\mu\;)B_\nu{}^\dagger \quad , \tag{2.4}$$

it follows from eq. (1.1) that

$$\begin{aligned} A'_\mu &= \frac{i}{g}B_\nu(B_\mu{}^\dagger\partial_\mu B_\mu)B_\nu{}^\dagger + \frac{i}{g}\,B_\nu\partial_\mu B_\nu{}^\dagger \\ &= \frac{i}{g}\,B_{\nu\mu}\partial_\mu B_{\mu\nu} \end{aligned} \tag{2.5}$$

where we have introduced the notation

$$B_{\mu\nu} \equiv B_\mu B_\nu{}^\dagger = B^\dagger_{\nu\mu} \quad . \tag{2.6}$$

Thus, repeating the argument for all μ and ν , we find

$$F_{\mu\nu} = \frac{i}{g} B_\nu{}^\dagger \partial_\nu (B_{\nu\mu} \partial_\mu B_{\mu\nu}) B_\nu \quad . \tag{2.7}$$

This result can of course be arrived at directly by substituting (1.1) in (2.2). The antisymmetry in μ and ν , although not manifest, is still not lost. This can be seen by considering

$$F_{\mu\nu} = B_\mu{}^\dagger F''_{\mu\nu} B_\mu \quad , \tag{2.8}$$

where $F''_{\mu\nu}$ is defined in the $A_\mu = 0$ gauge, and proceeding in a similar manner. Then

$$F_{\mu\nu} = -\frac{i}{g} B_\mu{}^\dagger \partial_\mu (B_{\mu\nu} \partial_\nu B_{\nu\mu}) B_\mu = -F_{\nu\mu} \quad , \tag{2.9}$$

as expected. Note that the variable $B_{\mu\nu}$ defined in eq. (2.6) can now be interpreted as a gauge transformation from the $A_\mu = 0$ gauge to the $A_\nu = 0$ gauge. This follows from eqs. (2.3) and (2.8), whence

$$F''_{\mu\nu} = B_{\mu\nu} F'_{\mu\nu} B_{\nu\mu} = B^\dagger_{\nu\mu} F'_{\mu\nu} B_{\nu\mu} \quad . \tag{2.10}$$

Thus, using eqs. (2.7) or (2.9), the Lagrangian eq. (2.1) becomes

$$L = \frac{1}{2g^2} \mathrm{Tr}\left[B_\mu{}^\dagger (\partial_\mu (B_{\mu\nu} \partial_\nu B_{\nu\mu})) B_\mu B^{\dagger\mu} (\partial^\mu (B^{\mu\nu} \partial^\nu B^{\nu\mu})) B^\mu \right] . \tag{2.11}$$

Since A_μ is bilinear in B_μ, we lose no generality in taking

$B^\mu = B_\mu$, provided we adhere to the usual diagonal Minkowski metric $\eta_{\mu\nu}$. Then the minus sign on the right-hand side of $A^I = -A_I$, $I = 1,2,3$ comes from $\partial^I = -\partial_I$. This will not be possible in gauges such as the lightcone gauge, where one must work with an off-diagonal metric, and $B^+ = B_- \neq B_+$ follows from $A^+ = A_- \neq A_+$. More generally in any basis with non-trivial metric B^μ has to be defined relative to B_μ as mentioned in the introduction.

Hence for the diagonal Minkowski metric, $B_\mu B^{\mu\dagger} = B_\mu B_\mu{}^\dagger = 1$ (no sum), and the Lagrangian simplifies to

$$L = \frac{1}{2g^2} \mathrm{Tr}\left[\partial_\mu (B_{\mu\nu}\partial_\nu B_{\nu\mu})\partial^\mu (B^{\mu\nu}\partial^\nu B^{\nu\mu})\right] \quad . \tag{2.12}$$

The theory is now written entirely in terms of the variables $B_{\mu\nu}$, which are gauge invariant by eqs. (1.4) and (2.6). In fact L contains time derivatives of B_{OI}, I=1,2,3 only, so that the B_{OI}'s can be regarded as the fundamental dynamical variables. The remaining $B_{\mu\nu}$,s are related to B_{OI} via the constraint

$$B_{IJ} = B_{IO}B_{OJ} \quad , \tag{2.13}$$

which follows from the definitions of $B_{\mu\nu}$(eq. (2.6)).

Of course there must be additional constraints which relate the B_{OI}'s or their conjugate momenta, as is dictated by the generalized Gauss' law which follows from gauge invariance. To find these constraints we must identify the canonical momentum conjugate to $B_{OI}^{(ij)}$, which we denote by

$\Lambda_{IO}^{(ji)}$. This is done by varying the action $S = \int d^4x L$ with respect to $\partial_0 B_{OI}^{(ij)}$. Integrating by parts, we find

$$\delta S = \frac{-1}{g^2} \int d^4x \; \mathrm{Tr}\left[\delta(\partial_0 B_{OI})\partial_I{}^2(B_{IO}\partial_0 B_{OI})B_{IO}\right] \quad ,$$

and therefore

$$\Lambda_{IO} = \frac{1}{g^2}\partial_I{}^2(B_{IO}\partial_0 B_{OI})B_{IO} \quad . \tag{2.14}$$

The equations of motion for B_{OI} follow in a similar manner by setting the variation with respect to B_{OI} equal to zero. Multiplying the equation of motion from the left by B_{OI} and summing over I=1,2,3, we obtain the relation

$$\partial_0\left(\sum_{I=1}^{3} B_{OI}\Lambda_{IO}\right) = 0 \quad , \tag{2.15}$$

or equivalently

$$\sum_{I=1}^{3} B_{OI}\Lambda_{IO} = G_0 \quad , \tag{2.16}$$

where G_0 is some time independent matrix which is determined by the boundary conditions, that is, by the properties of the vacuum. It is clear that eq. (2.16) is a constraint on the dynamical variables whose existence is directly related to the gauge invariance denoted by T_0 in eqs. (1.5, 1.6). This constraint is identical to Gauss' law

$$D_I E_I = 0 \quad , \tag{2.17}$$

as can be seen by rewriting eq. (2.17) in terms of our variables. It is then clear that we should demand $G_0 = 0$ on the physical gauge invariant states. Actually, in terms of our gauge invariant variables the constraint is easily imposed as an operator condition, as we will argue in the next section.

III. GAUGE INVARIANT HAMILTONIAN AND QUANTIZATION

From the Lagrangian of eq. (2.12), we obtain the symmetric, gauge invariant energy-momentum density,

$$\theta_{\mu\nu} = \frac{2}{g^2} \operatorname{Tr}\left[\partial_\alpha (B_{\alpha\mu}\partial_\mu B_{\mu\alpha})\partial^\alpha (B^\alpha{}_\nu \partial_\nu B_\nu{}^\alpha)\right] - g_{\mu\nu} L . \quad (3.1)$$

From this we obtain the Hamiltonian density,

$$\begin{aligned} \theta_{00} \equiv H &= -\frac{2}{g^2} \operatorname{Tr}\sum_I \left[\partial_I (B_{IO}\partial_0 B_{OI})\right]^2 - L \\ &= -\frac{1}{g^2} \operatorname{Tr}\sum_I \left[\partial_I (B_{IO}\partial_0 B_{OI})\right]^2 \\ &\quad -\frac{1}{g^2} \operatorname{Tr}\sum_{I>J} \left[\partial_I (B_{IJ}\partial_J B_{JI})\right]^2 \quad . \end{aligned} \quad (3.2)$$

Note that this is indeed positive definite, since $(B_{IJ}\partial_J B_{JI})^\dagger = -B_{IJ}\partial_J B_{JI}$. In order to express H in terms of the Λ_{IO}, we must solve for $\partial_I(B_{IO}\partial_0 B_{OI})$ in terms of Λ_{IO}:

$$\partial_I (B_{IO}\partial_0 B_{OI}) = g^2 \frac{1}{\partial_I} (\Lambda_{IO} B_{OI}) \quad , \quad (3.3)$$

where we define

$$\frac{1}{\partial_I} f(x^I) \equiv -\int_{x^I}^{\infty} dx^{I'} \; f(x^{I'}) \quad (3.4)$$

for some function f of x^I. Thus

$$H = -g^2 \ \mathrm{Tr} \sum_I \left[\frac{1}{\partial_I} (\Lambda_{IO} B_{OI})\right]^2 - \frac{1}{g^2} \mathrm{Tr} \sum_{I>J} \left[\partial_I (B_{IJ} \partial_J B_{JI})\right]^2 . \tag{3.5}$$

The second term in H is just the usual magnetic term, $\sum_{I>J} F_{IJ}{}^2$. Similarly, the first term is related to the usual electric field term $\sum_I E_I{}^2$, although its significance is more transparent in the light of the commutation rules, to which we now turn our attention.

Since $\Lambda_{IO}^{(ji)}$ is conjugate to $B_{OI}^{(ij)}$, one may be tempted to take the naive commutation rule,

$$\left[\Lambda_{IO}^{(ij)}(x),\ B_{OJ}^{(lm)}(x')\right] = -i\delta^{im}\delta^{jl}\delta_{IJ}\delta^3(x-x'). \tag{3.6}$$

Recall, however, that the relationship between A_μ and B_μ is subject to the constraints, det $B_\mu = 1$, Tr $B_\mu^\dagger \partial_\mu B_\mu = 0$, $B_\mu^\dagger B_\mu = 1$. Translated into the language of B_{OI}'s and Λ_{IO}'s, these become a set of 2nd class constraints (ignoring for the moment $B_{IO}B_{OI}=1$):

$$\det B_{OI} = 1 \tag{3.7}$$

$$\mathrm{Tr}\ \Lambda_{IO} B_{OI} = 0 \quad . \tag{3.8}$$

Note that had we considered an U(N) theory instead of SU(N), we would not have such constraints. These constraints are not consistent with the naive commutation rules (3.6), and they are not easily solved explicitly. We must therefore resort to the method of Dirac brackets[6] in order to find the correct commutation rules. In addition, we must deal with the constraints(2.16) and $B_{IO}B_{OI}=1$. We will return to these after we modify the commutation rules to first take care of (3.7) and (3.8).

The Dirac brackets are constructed as follows. We start with the naive Poisson brackets,

$$\{\Lambda_{IO}^{(ij)}(x),B_{OI}^{(lm)}(x')\} = \delta^{im}\delta^{jl}\delta^{3}(x-x') \quad . \tag{3.9}$$

The constraints are

$$\phi_{I}^{(a)}(x) = 0 \quad , \tag{3.10}$$

where

$$\phi_{I}^{(1)} = \det B_{OI} - 1 \quad , \tag{3.11a}$$

$$\phi_{I}^{(2)} = \mathrm{Tr}\,\Lambda_{IO}B_{OI} \quad . \tag{3.11b}$$

Define the matrix M_I constructed through the naive Poisson brackets (3.9)

$$\delta_{IJ}M_{I}^{ab}(x,x') = \{\phi_{I}^{a}(x),\phi_{J}^{b}(x')\} \quad . \tag{3.12a}$$

Then the Dirac brackets are given by

$$\{\Lambda_{IO}^{(ij)}(x),B_{OI}^{(lm)}(x')\}^{*} = \{\Lambda_{IO}^{(ij)}(x),B_{OI}^{(lm)}(x')\}$$

$$- \int d^{3}y\,d^{3}z\,\{\Lambda_{IO}^{(ij)}(x),\phi_{I}^{a}(y)\}(M_{I}^{-1})^{ab}(y,z)\{\phi_{I}^{b}(z),B_{OI}^{(lm)}(x')\} \quad . \tag{3.12b}$$

Using equations (3.9) - (3.12), we find that the only nonzero element of M_{I}^{ab} is

$$M_I^{12}(x,x') = -M_I^{21}(x',x) = -N\delta^3(x-x') \quad . \tag{3.13}$$

then

$$(M_I^{-1})^{21}(x',x) = -\frac{1}{N}\delta^3(x'-x) \quad . \tag{3.14}$$

In order to evaluate eq. (3.12) we also need:

$$\begin{aligned} \{\Lambda_{IO}^{(ij)}(x), \phi_I^{(1)}(x')\} &= \delta^3(x-x')B_{IO}^{(ij)} \\ \{\phi_I^{(2)}(x'), B_{OI}^{(lm)}(x)\} &= B_{OI}^{(lm)}\delta^3(x-x') \quad . \end{aligned} \tag{3.15}$$

we finally obtain

$$\{\Lambda_{IO}^{(ij)}(x), B_{OI}^{(lm)}(x')\}^* = (\delta^{im}\delta^{jl} - \frac{1}{N}\, B_{IO}^{(ij)}B_{OI}^{(lm)})\delta^3(x-x') \quad . \tag{3.16}$$

The commutators of the quantized theory are given by the correspondence $\{,\ \}^* \to i\Big[,\Big]$. Thus we find

$$\Big[\Lambda_{IO}^{(ij)}(x), B_{OI}^{(lm)}(x')\Big] = -i(\delta^{im}\delta^{jl} - \frac{1}{N}B_{IO}^{(ij)}B_{OI}^{(lm)})\delta^3(x-x') \quad . \tag{3.17}$$

Note that the Λ_{IO}'s also do not commute. However, $\Lambda^{(ij)},\Lambda^{(lm)}$ is not very illuminating, since Λ_{IO} and B_{OI} occur in the Hamiltonian only in the combination $\Lambda_{IO}B_{OI}$, and it is this operator which is significant. Also note that we have not yet taken the constraint eq. (2.16) and $B_{IO}B_{OI} = 1$ into account.

Let us now consider the operator

$$G_I(x) \equiv \frac{i}{2}\,\Lambda_{IO}(x)B_{OI}(x) \quad . \tag{3.18}$$

Since $\Lambda_{IO}B_{OI}$ is traceless and hermitian (See 2.14), G_I can be written as:

$$G_I(x) = \frac{\lambda^a}{2} G_I^a(x) \quad , \qquad (3.19)$$

where the $\lambda^a/2$'s ($a=1\ldots N^2-1$) are the traceless $N\times N$ representations of the generators of SU(N). They obey the identity

$$\tfrac{1}{2}(\lambda^a)_{ij}(\lambda^a)_{lm} = \delta_{im}\delta_{jl} - \frac{1}{N}\,\delta_{ij}\delta_{lm} \quad . \qquad (3.20)$$

Using eqgs. (3.17) - (3.20), it is easy to show that

$$\left[G_I^a(x), B_{OJ}(x')\right] = \left(B_{OJ}\,\frac{\lambda^a}{2}\right)\delta^3(x-x')\,\delta_{IJ}. \qquad (3.21)$$

Taking the hermitian conjugate of this equation, we find

$$\left[G_I^a(x), B_{JO}(x')\right] = -\,\frac{\lambda^a}{2} B_{JO}\,\delta^3(x-x')\,\delta_{IJ} \quad . \qquad (3.22)$$

It can now be checked that (3.21) and (3.22) are consistent with $B_{IO}B_{OI} = 1$. In fact from these one can deduce the correct commutation rules between Λ_{IO} and $B_{JO} = B_{OJ}^{\dagger}$, which together with (3.17) are consistent with this constraint.

The Hamiltonian can now be expressed entirely in terms of the G_I's and B_{IJ}'s as seen from eqs. (3.5) and (3.18). These variables are invariant not only under the original gauge transformation (1.4) but also under the $T_0(x)$ type gauge transformation of eq. (1.6), as is obvious from eq. (2.13) and (3.3). The quantum properties of the theory are now given by the commutation rules of these gauge invariant variables, which can be derived from eqs. (3.21), (3.22) and (2.13):

$$\left[G_I^a(x),\ B_{JK}(x')\right] = \{-\ \delta_{IJ}\frac{\lambda^a}{2} B_{JK} + \delta_{IJ}B_{JK}\frac{\lambda^a}{2}\}\ \delta(\vec{x}-\vec{x}') \tag{3.23}$$

The commutators of G_I^a can be calculated directly using the Dirac formalism of eqs. (3.10-3.14), or by commuting both sides of eq. (3.23) with G_L^b and applying the Jacobi identity. In either case, we obtain

$$\left[G_I^a(x),\ G_J^b(x')\right] = i\ f^{abc}\ G^c(x)\delta(\vec{x}-\vec{x}')\delta_{IJ} \tag{3.24}$$

These commutation rules indicate that the $G_I(x)$ act like the generators of a local $SU(N)_1 \otimes SU(N)_2 \otimes SU(N)_3$ group, which is realized on the $B_{IJ}(x)$. They are reminiscent of the commutation rules that one encounters in the study of the non-linear sigma model. We remind the reader that in the non-linear $SU(N) \otimes SU(N)$ sigma model one deals with left-handed and right-handed transformations on a unitary matrix Σ. If one denotes the generators of these left(L) and right (R)-handed transformations by G_L^a and G_R^a one finds a constraint of the form

$$G_L + :\Sigma\ G_R\Sigma^\dagger: = 0 \quad . \tag{3.25}$$

In our case there are 3 unitary matrices B_{12}, B_{13}, B_{23} and 3 generators G_1^a, G_2^a, G_3^a which act on the 1,2,3 "sides" of the B's as in eq. (3.23). Analogous to the constraint (3.25) of the sigma model, we have the constraint $G_0 = 0$ as given in eq.

(2.16) which now takes the form[†] (via 3.18):

$$\sum_{I=1}^{3} : B_{0I} G_I B_{I0} : = G_0 \quad . \tag{3.26}$$

It can be checked by using the commutation rules (3.21) and (3.22) that G_0^a is just the generator of the gauge transformation $T_0(\vec{x})$ of eq. (1.6). Therefore, it must commute with our gauge invariant variables G_I^a and B_{IJ}. This indeed is true and can be verified by direct commutation. Since the theory is already expressed only in terms of singlet operators we can set $G_0{}^a = 0$, satisfying Gauss' law identically.

We have to ask now whether our commutation rules (3.23) and (3.24) are consistent with this contraint. The constraint can easily be solved as follows: We apply to eq. (3.26) a unitary transformation in the form $:B_{J0} G_0 B_{0J}: = 0$ for any J, to obtain

$$\sum_{I=1}^{3} :B_{JI} G_I B_{IJ}: = 0 \quad (\text{any } J) \quad . \tag{3.27}$$

[†] In (3.26) :(...): implies an ordering of operators to insure tracelessness of G_0 and consistency with the commutation rules. The ordering is defined as[1]

$$:B^\dagger G\, B: \equiv G^a B^\dagger \frac{\lambda^a}{2} B = B^\dagger \frac{\lambda^a}{2} B\, G^a \quad .$$

It is seen that $:B^\dagger G\, B:$ is traceless, hermitian and consistent with the commutation rules (3.21) and (3.22). If one insists in placing G^a in between $B^\dagger$ and B, a c-number term proportional to the matrix 1 must be added to be consistent with these properties.

We also remember that the B_{IJ} satisfy by construction (see eq. (2.6) or eq. (2.13))

$$B_{IJ}B_{JK} = B_{IK} \quad (\text{any } J) \quad . \tag{3.28}$$

This implies that one can choose just two B's and two G's as being independent and solve for the 3rd B and G through eqs. (3.27) and (3.28). For example if we take J=3, then

$$B_{12} = B_{13}B_{32} \tag{3.29}$$

$$G_3 = -\left[:B_{31}G_1B_{13}: + :B_{32}G_2B_{23}\right] \quad . \tag{3.30}$$

Eq. (3.30) is a generalization of eq. (3.25) of the sigma model. We can now check the commutation properties of B_{12} and G_3 by treating them as dependent variables as in (3.29) and (3.30). The remarkable property is that they continue to satisfy the basic commutation rules (3.23) and (3.24). Since the operator properties of our variables are insensitive to the choice of dependent and independent variables, we will continue to preserve the symmetry between the 3 axes I=1,2,3 by keeping the notation as before, bearing in mind that the constraint is automatically satisfied.

We have argued that our variables G_I^a and B_{IJ} are gauge invariant with respect to the gauge transformations $U(x_\mu)$ and $T_0(\vec{x})$ of eqs. (1.4) and(1.6). There remains the gauge transformations T_I I=1,2,3. It is easy to verify by direct (equal time) commutation that the generators Q_I^a of these transformations are the complete integrals of G_I^a in the x_I variable for each I. That is

$$Q_1 = \int_{-\infty}^{\infty} G_1(x_1x_2x_3)dx_1;\ Q_2 = \int_{-\infty}^{\infty} G_2(x_1x_2x_3)dx_2;\ Q_3 = \int_{-\infty}^{\infty} G_3(x_1x_2x_3)dx_3 \ . \tag{3.31}$$

The physical states must be annihilated by these Q_I^a, since they must be gauge invariant. We refer to refs. 1 and 5 for a discussion of this point. Briefly, the Q_I^a are proportional to the electric field at ∞ (see eq. (3.32) below), and for that reason the gauge non-invariant unphysical states, only on which Q_I^a is non-zero, acquire an ∞ energy and disappear from the spectrum automatically. In the strong coupling approach this condition is easy to implement and restricts the spectrum only to closed strings or strings with quarks at the ends.

Finally to make contact with ref. 1 we express the Hamiltonian in terms of the variables

$$\Pi_I(\vec{x}) = -\frac{1}{\partial_I} G_I(x) = \int_{x^I}^{\infty} dx^{I'} G(x^{I'}) \quad . \tag{3.32}$$

The Π_I for given I can be interpreted as the electric field in the $A_I = 0$ gauge.[1] Thus, the Hamiltonian written in terms of gauge invariant variables is

$$H = \mathrm{Tr}\{ g^2 \sum_I (\Pi_I)^2 = \frac{1}{g^2} \sum_{I>J} \partial_I (B_{IJ}\partial_J B_{JI})^{\ 2} \quad . \tag{3.33}$$

This is the theory used in refs. 1,2 as described in the introduction. We have given here a method for deriving the quantization rules of this theory, and simultaneously we have shown that the variables are gauge invariant.

IV. LIGHT CONE QUANTIZATION

Lagrangian, Equations of Motion, and Constraints

As an example of how this formalism works for a basis which requires an off-diagonal metric, we explore the light-cone basis. We take $x^+ = \frac{1}{\sqrt{2}}(x^0 - x^3)$ to be the "time variable, and $x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$, $x_i (i=1,2)$ as "space" variables. On the light cone, eq. (1.1) is replaced by

$$\begin{aligned} A_\pm &= \frac{i}{g} B_\pm^\dagger \partial_\pm B_\pm \quad , \\ A^\pm &= \frac{i}{g} B^\pm \partial^\pm B^\pm = A_\mp \\ A_i &= \frac{i}{g} B_i^\dagger \partial_i B_i = -A^i \quad . \end{aligned} \tag{4.1}$$

Although the A_μ-variables are simply related to those used previsouly (e.g., $A_+ = \frac{1}{\sqrt{2}}(A_0 + A_3)$), this is not so for the B_μ's. The relationship between $B_\pm$ on the one hand and B_0 and B_3 on the other is complicated and nonlinear. However, the substitution of eq. (4.1) is the most natural generalization of eq. (1.1).

The derivation of the Lagrangian goes through exactly as before, with the same interpretation for the $B_{\mu\nu}$'s (e.g., B_{+-} is a gauge transformation from the $A_+ = 0$ gauge to the $A_- = 0$ gauge). We must be careful, however, to use eq. (2.11) rather than (2.12), since because of the off-diagonal metric $B^+ = B_-$ and $B^- = B_+$. Thus $B_+ B^{+\dagger} = B_{+-}$ rather than one. Then using eq. (2.11) we have

$$L = \frac{1}{2g^2}\int d^3x \, \mathrm{Tr}\Big[B_{+-}\partial_-(B_{-+}\partial_+B_{+-})B_{-+}\partial_+(B_{+-}\partial_-B_{-+})$$
$$+ B_{-+}\partial_+(B_{+-}\partial_-B_{-+})B_{+-}\partial_-(B_{-+}\partial_+B_{+-})$$
$$-4\sum_i \partial_i(B_{i-}\partial_-B_{-i})\partial_i(B_{i+}\partial_+B_{+i})$$
$$+ 2\partial_1(B_{12}\partial_2B_{21})\partial_1(B_{12}\partial_2B_{21})\Big] \quad .$$

Using the identity

$$B_{-+}\partial_+(B_{+-}\partial_-B_{-+})B_{+-} = -\partial_-(B_{-+}\partial_+B_{+-}) \quad , \qquad (4.2)$$

this becomes:

$$L = \frac{1}{g^2}\int d^3x \, \mathrm{Tr}\Big[-(\partial_-(B_{-+}\partial_+B_{+-}))^2$$
$$-2\sum_i \partial_i(B_{i-}\partial_-B_{-i})\partial_i(B_{i+}\partial_+B_{+i})$$
$$+ (\partial_1(B_{12}\partial_2B_{21}))^2\Big] \quad . \qquad (4.3)$$

Once again, L is expressed entirely in terms of $B_{\mu\nu}$'s. x^+ (i.e., "time") derivatives occur only for B_{+-} and B_{+i}, so we take these as dynamical variables. Proceeding in an analogous fashion to that of the previous section, we find that the conjugate momentum to B^{ij}_{+-}, denoted by Λ^{ji}_{-+} . is

$$\Lambda_{-+} = \frac{2}{g^2}\,\partial_-{}^2(B_{-+}\partial_+B_{+-})B_{-+} \quad . \qquad (4.4)$$

Similarly, the conjugate momentum of B^{ij}_{+i}, denoted by Λ^{ji}_{i+}, is

$$\Lambda_{i+} = \frac{2}{g^2}\partial_i{}^2(B_{i-}\partial_-B_{-i})B_{i+} \quad . \qquad (4.5)$$

Note that eq. (4.5), since it doesn't contain time derivatives, takes the form of a constraint, and must be dealt with by the Dirac bracket method.

The Gauss' law constraint is found from the equations of motion for B_{+-} and B_{+i} as before. We obtain

$$\partial_+(B_{+-}\Lambda_{-+} + \sum_i B_{+i}\Lambda_{i+}) = 0 \quad . \tag{4.6}$$

Thus

$$B_{+-}\Lambda_{-+} + \sum_i B_{+i}\Lambda_{i+} = G_+ \quad , \tag{4.7}$$

where G_+ is time-independent. We will find, once again, that G_+^a is the generator of T_+-type (i.e., x^+-independent) transformations on the "+" side of B_{+-}, B_{+i}, etc., and that only B_{12} and G_- appear in the Hamiltonian. It will then be possible to set $G_+ = 0$ for all "time".

Hamiltonian and Quantization

We proceed to find $\theta_{\mu\nu}$ as in Section II, however we must now be careful to use the Lagrangian of eq. (2.11). Then

$$\begin{aligned}\theta_{\mu\nu} = {} & \frac{1}{g^2}\,\mathrm{Tr}(\partial_\alpha(B_{\alpha\mu}\partial_\mu B_{\mu\alpha})B_\alpha{}^\alpha\partial^\alpha(B^\alpha{}_\nu\partial_\nu B_\nu{}^\alpha)B^\alpha{}_\alpha) \\ & + \frac{1}{g^2}\,\mathrm{Tr}(\partial_\mu(B_{\mu\alpha}\partial_\alpha B_{\alpha\mu})B_{\mu\nu}\partial_\nu(B_\nu{}^\alpha\partial^\alpha B^\alpha{}_\nu)B_{\nu\mu}) - g_{\mu\nu}L \quad .\end{aligned} \tag{4.8}$$

The Hamiltonian density is defined as usual as $\theta_+{}^+$, or

$$\theta_{+-} = - \operatorname{Tr}\left[g^2\left[\frac{1}{\partial_-} (\Lambda_{-+}B_{+-})\right]^2 + \frac{1}{g^2} (\partial_1(B_{12}\partial_2 B_{21}))^2\right] \quad . \tag{4.9}$$

Thus all terms involving Λ_{i+} and $B_{\pm i}$ have canceled. It is in fact a well-known result that only F_{+-} and F_{12} occur in the Hamiltonian in any lightcone formulation. The simplicity of the Hamiltonian is one of our main motivations for exploring the light cone gauge. We will see, however, that this simplicity is achieved at the expense of complicated commutation rules.

The complications arise in the two extra constraints which must agree with the commutation rules,

$$G_i = \frac{-i}{g^2} \partial_i{}^2(B_{i-}\partial_- B_{-i}) \quad , \tag{4.10a}$$

where we have defined

$$G_i = \frac{-i}{2} \Lambda_{i+} B_{+i} \quad . \tag{4.10b}$$

We already know, by the method of the previous section, how to deal with the constraints

$$\det B_{+i} = 0$$

$$\operatorname{Tr} \Lambda_{i+} B_{+i} = 0 \quad . \tag{4.11}$$

Thus in applying the Dirac bracket method the constraints (4.11) are easily taken care of: Before dealing with (4.10) we can use (with k,n = 1,2,−)

$$\{\Lambda_{k+}^{(ij)}(x),\ B_{+n}^{(lm)}(x')\} = (\delta^{im}\delta^{jl} - \frac{1}{N} B_{k+}^{(ij)}B_{+n}^{(lm)})\delta_{kn}\delta^3(x-x') \tag{4.12}$$

as the "naive" Poisson brackets. At this point all of the statements of Section III after eq. (3.16) hold. The G_i^a's act as generators of $SU(N)_i$ transformations on the i-side of $B_{\pm i}$ just as in eqs. (3.21-22), and similarly

$$G_-^a \equiv -i\ \mathrm{Tr}\frac{\lambda^a}{2}\ \Lambda_{-+}B_{+-} \tag{4.13}$$

generates such transformations on the "-" side of B_{-i} and B_{-+}. The quantity G_+ defined in eq. (3.7) has zero Poisson bracket with all the other G's, B_{-i} and B_{12} (and hence with the Hamiltonian). Thus once again we need not incorporate eq. (4.7) and the unitarity conditions $B^+B = 1$ in the Dirac bracket formalism, but we must deal with the new constraints (4.4) or equivalently (4.10) that arise on the lightcone.

As can be seen from eq. (4.9), only $\Pi_- \equiv \frac{-1}{\partial_-} G_-$ and B_{12} occur in the Hamiltonian,

$$H = \theta_{+-} = \mathrm{Tr}\left(g^2\Pi_-^2 - \frac{1}{g^2}\,(\partial_1(B_{12}\partial_2B_{21}))^2\right)\ . \tag{4.14}$$

Therefore, to illustrate the method, we calculate the Dirac brackets (and hence commutators) for these variables $\{\Pi_-^a,\ B_{12}\}^*$ and $\{B_{12}^{(ij)}(x),\ B_{21}^{(lm)}(x')\}^*$. Unfortunately, these commutators are found not to be very useful due to their complexity, and are much more difficult to interpret than those of Section III.

The constraints eqs. (4.10) can be written in the form

$$\phi_i^a \equiv G_i{}^a + \frac{i}{g^2}\,\partial_i{}^2\ \mathrm{Tr}\left(\frac{\lambda^a}{2}\ B_{i-}\partial_- B_{-i}\right) = 0 \quad . \qquad (4.15)$$

Although Π_- has simple Poisson brackets with the left-hand side of this equation, the matrix $M_{ij}^{ab}(x,x')$, (eq. (3.13)) that results from this constraint is impossible to invert explicitly A more tractable form of eq. (4.15) which yields an easily invertible form for $M_{ij}^{ab}(x,x')$ is (no sum on i).

$$B_{-i}\Pi_i B_{i-} + \frac{i}{g^2}\,\partial_-(B_{-i}\partial_i B_{i-}) = 0 \quad . \qquad (4.16)$$

where Π_i is defined as in Section III, and we have used an identity similar to eq. (4.2). Then defining new

$$\phi_i{}^a = \mathrm{Tr}\ \frac{\lambda^a}{2}\left(B_{-i}\Pi_i B_{i-} + \frac{i}{g^2}\partial_-(B_{-i}\partial_i B_{i-})\right) \ , \qquad (4.17)$$

and noting that

$$\{\mathrm{Tr}\ \frac{\lambda^a}{2}\ B_{-i}\Pi_i B_{i-},\ \mathrm{Tr}\ \frac{\lambda^b}{2}\ B_{-j}\Pi_j B_{j-}\} = 0 \qquad (4.18)$$

we find

$$\begin{aligned}\{\phi_i{}^a(x),\ \phi_j{}^b(x')\} &= -\frac{1}{g^2}\ \delta^{ab}\partial_-\delta^3(x-x')\delta_{ij}\\ &\equiv M^{ab}(x-x')\delta_{ij} \quad . \qquad (4.19)\end{aligned}$$

Because of eq. (4.18), only the cross terms have contributed to the resulting eq. (4.19). Thus from eq. (4.19),

$$(M^{-1})^{ba}(x'x) = \frac{-g^2}{2}\delta^{ab}\varepsilon(x'_- - x_-)\delta(x'_1 - x_1)\delta(x'_2 - x_2) \quad . \tag{4.20}$$

The ε-symbol is used as the inverse of ∂_- to insure the correct symmetry properties for the Dirac brackets. We also need the following to calculate the Dirac brackets:

$$\{\Pi_-{}^a,\ \phi_i{}^b(x')\} = \frac{1}{2g^2}\,\delta^{ab}\partial_i\delta^3(x-x') - \frac{i}{g^2}\,f^{abc}\ \mathrm{Tr}\frac{\lambda^c}{2}\,B_{-i}\partial_i B_{i-}\delta^3(x-x') \tag{4.21}$$

$$\{\phi_1{}^a(x), B_{12}(x')\} = i\theta(x'_1 - x_1)\delta(x'_2 - x_2)\delta(x'_- - x_-) \times B_{1-}\,\frac{\lambda^c}{2}\,B_{-1}B_{12}(x') \quad , \tag{4.22}$$

$$\{\phi_2{}^a(x), B_{12}(x')\} = -i\theta(x'_2 - x_2)\delta(x'_1 - x_1)\delta(x'_- - x_-) \times B_{12}(x')B_{2-}\,\frac{\lambda^a}{2}\,B_{-2} \quad , \tag{4.23}$$

$$\{\Pi_-{}^a,\ B_{12}(x')\} = 0 \quad , \tag{4.24}$$

$$\{B_{12}^{(ij)},\ B_{21}^{(lm)}(x')\} = 0 \quad . \tag{4.25}$$

Then we find

$$\{\Pi_-{}^a(x),\ B_{12}(x')\}^* =$$

$$- \tfrac{1}{2}\theta(x'_1 - x_1)\varepsilon(x_- - x'_-)\delta(x_2 - x'_2)f^{abc}\ \mathrm{Tr}\ \frac{\lambda^c}{2}\,B_{-1}(x)\partial_1 B_{1-}(x)$$
$$\times B_{1-}(x_1 x'_2 x'_-)\ \frac{\lambda^b}{2}\ B_{-1}(x_1 x'_2 x'_-)B_{12}(x'_1 x'_2 x'_-)$$
$$+ \tfrac{1}{2}\theta(x'_2 - x_2)\varepsilon(x_- - x'_-)\delta(x_1 - x'_1)f^{abc}\ \mathrm{Tr}\ \frac{\lambda^c}{2}\,B_{-2}(x)\partial_2 B_{2-}(x)$$

(cont.)

$$\times B_{12}(x_1'x_2'x_-')B_{2-}(x_1'x_2x_-')\frac{\lambda^b}{2}B_{-2}(x_1'x_2x_-') \quad , \qquad (4.26)$$

and

$$\{B_{12}^{(ij)}(x),B_{21}^{(lm)}(x')\}^* = \frac{g^2}{2}\,\varepsilon(x_- - x_-')\delta(x_2-x_2')\int dy_1\,\theta(x_1-y_1)\theta(x_1'-y_1)$$
$$\times\Big[[B_{1-}(y_1x_2x_-)B_{-1}(y_1x_2x_-')]^{(im)}[B_{21}(x')B_{1-}(y_1x_2x_-')B_{-1}(y_1x_2x_-)$$
$$\times B_{12}(x)]^{(lj)} - 2B_{12}^{(ij)}(x)B_{21}^{(lm)}(x')$$
$$+ [B_{12}(x)B_{2-}(y_1x_2x_-)B_{-2}(y_1x_2x_-')B_{21}(x')]^{(im)}[B_{2-}(y_1x_2x_-')$$
$$\times B_{-2}(y_1x_2x_-)]^{(lj)}\Big] \quad . \qquad (4.27)$$

Unfortunately it is not at all clear from these equations what the commutators of the quantum theory are, since the strings of $B_{\mu\nu}$ operators on the right-hand sides present numerous ordering problems. The only operator which retains simple commutation rules like those of Section III is $G_+{}^a$, since it commutes with the constraints, eq. (4.17). Hence whatever the quantum versions of eqs. (4.26) and (4.27) are, they are still consistent with the constraint $G_+ = 0$, and insensitive to the choice of dependent variable (B_{1-}, B_{2-}, or B_{12}).

We further note that generalization to more complicated metrics (e.g., that of the u-basis of eq. (1.8)) will not necessarily present the same kind of difficulties as those of the lightcone. The complexity of eqs. (4.26) and (4.27) are a result of our choice of time, x^+, which induces two extra constraints. This can always be avoided if we quantize only on spacelike surfaces.

We have shown that the quantum algebra of the new variables for QCD, introduced in ref. 1, can be reproduced from a more fundamental Lagrangian approach. Furthermore, the dynamical variables are completely gauge invariant. It is found that the generalized Gauss' law, which is readily obtained from the equations of motion, can be imposed as an operator constraint since it commutes with all variables occuring in the Hamiltonian. The commutation rules are insensitive to the choice of dependent variables. Quantization on the light cone leads to certain difficulties which arise because of the choice of time, although it is possible that a more judicious choice of variables may circumvent these problems.

ACKNOWLEDGMENTS

This work was completed while one of us (I.B.) was visiting Fermilab. He is grateful to the theory group for their warm hospitality.

REFERENCES

[1]I. Bars, Phys. Rev. Letters 40, 688 (1978); and Proceedings of the Orbis Scientiae, Coral Gables 1978, to be published.

[2]I. Bars, "A Map Between Corner and Link Operators in Lattice Gauge Theories," Yale Report C00-3075-195, to be published.

[3]K. Wilson, Phys. Rev. D10, 2445 (1974); J. Kogut and L. Suskind, Phys. Rev. D11, 395 (1975).

[4]W. Kummer, Acta Physica Austriaca 14, 149 (1961); R.L. Arnowitt and S. Fickler, Phys. Rev. 127, 1821 (1962); J. Schwinger, Phys. Rev. 130, 402 (1963); R.N. Mohapatra, Phys. Rev. D4, 2215 (1971); W. Konetschny and W. Kummer, Nucl. Phys. B100, 106 (1975); A. Chodos, "Canonical Quantization of Non-Abelian Gauge Theories in Axial Gauge, Yale Report C00-3075-183, to be published.

[5]I. Bars and Frederic Green, "QCD in the Axial Gauge,Boundary Terms and Poincaré Invariance," Yale Report C00-3075-191, to be published in Nuclear Physics.