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The Behavior of Homogeneous Turbulence Mixed at Long Wavelengths

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ABSTRACT

Using the renormalization group we treat the theory of stationary, homogeneous, isotropic turbulence of a fluid mixed by a random force which has its strength concentrated at small wave number. We find that for mixing which does not vanish at zero wave number that the effective Reynolds number always remains small in $D \neq 4$ space dimensions. At small wave number the energy spectral function $E(k)$ behaves as $k^{-\rho}$ at D=3 with $\rho = -(4-D)/3$ in an expansion about D=4. For large wave number $E(k)$ is entirely governed by the forcing function. We show in some detail how to construct for all k the two fold velocity correlation function using the one loop approximation for the renormalization group functions. This extends beyond perturbation theory for the correlation functions and is improvable in a systematic fashion.

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I. INTRODUCTION

For many years it has been clear that the theory of fully developed turbulence is closely connected with problems in quantum field theory.[1] This is first brought out by the Hopf equation for the generating functional for the velocity correlation functions. Work subsequent to Hopf has utilized this observation in a productive fashion to formulate the equations of turbulent motion in a manner amenable to non-perturbative approximations.[2]

In the first paper in this series[3] this connection was made quite explicit by observing that both turbulent motion and quantum field theory are examples of *stochastic field theories*. The measure of stochasticity in quantum field theory is of course $\hbar$. A similar stochasticity parameter, called a in Ref. 3, was introduced to give a sense to the extent or fullness of the stochastic behavior of the classical fluid system. This analogue was then pursued by exploring the consequences of a non-perturbative technique, *the renormalization group*, to study both the small and large wave number behavior of the velocity correlation functions for turbulence.

The systems we describe are homogeneous, isotropic and stationary. Furthermore we assume that the turbulent motion is maintained by an external force field which puts in energy to compensate for the losses through viscous dissipation.

Our essential result is an evaluation of the importance of the non-linearities in the Navier-Stokes equation. This is contained in the effective coupling constant or *effective Reynolds number* of the motion. This effective size of the non-linear effects depends on the part of wave number space one is in. For very large wave number, the effective Reynolds number approaches zero rapidly in less than four space dimensions. So non-linear effects are absent and the velocity field effectively tracks the random force field. In a sense this is not surprising since the viscous terms in the Navier-Stokes equation would obviously seem dominant at large wave number. However, the fact that for four or more space dimensions this "dimensional" argument fails means the situation is slightly more subtle.

What is most surprising is that for small wave number as well, for a large class of realistic forcing functions, the effective Reynolds number is still small. Perturbation theory is modified in a smooth, calculable fashion. Still the velocity tracks the forcing field to a large extent. However the piling up of singularities at small frequency and wave number eventually cuts off the spectrum in the small wave number region.

Batchelor[4] has described the problem of homogeneous turbulence as being the determination of the velocity field distribution at sometime given what it was at an earlier time; the dynamics to be governed, of course, by the Navier-Stokes equation. As emphasized by Novikov,[5] this is appropriate in the non-stationary problem while in the stationary problem, as treated in this paper only an average over the random forcing introduced to maintain the turbulence is needed. These forces F_j are akin to some velocity distribution because of the Navier-Stokes equation

$$F_j(\vec{x},t) = \frac{\partial}{\partial t} v_j - \nu\nabla^2 v_j + \tfrac{1}{2}[(\Delta_{jn}\nabla_\ell + \Delta_{j\ell}\nabla_n)(v_n v_\ell)] \tag{1}$$

with $\Delta_{j\ell} = \delta_{j\ell} - \nabla_j\nabla_\ell/\nabla^2$ coming from incompressibility. The random force allows us to treat a stationary problem where at $t \to -\infty$ the fluid was quiescent and the turbulence has been stimulated by the forcing. In the non-stationary problem we can consider a quiescent fluid which is stirred for some time by random forces and then permitted to decay. The initial velocity field distribution is established by the stirring and Batchelor's formulation now becomes the question of observing the decay.

We will take the random stirring force to be gaussian with zero mean, so all we need is the correlation function. To concentrate on the renormalization group aspects of the theory we studied earlier a wholly unrealistic forcing field which mixed the fluid uniformly in frequency and wave number. In this paper we examine in detail a more realistic story where the mixing is concentrated at small wave numbers $k \lesssim k_0$, or equivalently is peaked about some external scale k_0^{-1}. We deal still with the stationary problem so the mixing is always on.

To be more precise we consider a force field $F_j(\vec{x},t)$ which is solenoidal, has zero mean and correlation function

$$\langle F_j(\vec{x},t) F_\ell(\vec{y},\tau)\rangle = \Delta_{j\ell}(\nabla)\hat{\Gamma}(\vec{x}-\vec{y};t-\tau) \tag{2}$$

We shall take the forces to be δ-function correlated in time so

$$\hat{\Gamma}(\vec{x},t) = \tilde{\Gamma}(\vec{x})\delta(t) \quad . \tag{3}$$

Also we write

$$\tilde{\Gamma}(\vec{k}) = \frac{\gamma_0^2}{4}\tilde{\Gamma}(k^2/k_0^2) = \int d^Dx \, e^{i\vec{k}\cdot\vec{x}}\,\tilde{\Gamma}(\vec{x}) \quad , \tag{4}$$

where we work in D space dimensions, and all the dimensions of $\tilde{\Gamma}$ are contained in γ_0^2.

Now with such a correlation function, Novikov[5] has shown that the energy dissipation, $\mathcal{E}$, due to viscosity is

$$\mathcal{E} = \frac{(D-1)}{2}\int \frac{d^Dp}{(2\pi)^D}\, \Gamma_M(p^2/k_0^2)\frac{\gamma_0^2}{4} \tag{5}$$

$$= \frac{\gamma_0^2}{4}\, k_0^D\, \frac{(D-1)}{2}\int \frac{d^Dp}{(2\pi)^D}\, \Gamma_M(p^2) \quad . \tag{6}$$

In our earlier work we chose $\tilde{\Gamma}_M(p^2) = 1$. Of course, then $\mathcal{E}$ is infinite as one would expect on physical grounds. Here we certainly wish to require that $\mathcal{E}$ is finite, which means $\tilde{\Gamma}(0) \neq 0$ or that (5) and (6) converge. Very few of our general observations will depend on the detailed form of $\Gamma_M(k^2)$. It will matter whether $\Gamma_M(k^2 = 0)$ vanishes or not. For most examples of correlation functions peaked near $x \approx k_0^{-1}$, $\Gamma_M(k^2 = 0)$ is finite, so our attention will be turned to such behavior.

With an external scale in the problem rather little changes in the details of our analysis. We do have a natural length on which to base the Reynolds number. When $\Gamma_M(k^2) = 1$, then there is no external scale and the Reynolds number must be defined in terms of the renormalization scale k_N introduced in Ref. 3. According to the dimensional analysis done there, the only dimensionless quantity around is

$$\frac{\gamma_0}{\nu^{3/2}}(\text{length})^{(4-D)/2} \quad , \tag{7}$$

where ν is the kinematic viscosity of the fluid. The Reynolds number based on the external length k_0^{-1} is

$$R = \frac{\gamma_0}{\nu^{3/2}}\, k_0^{(D-4)/2} \quad . \tag{8}$$

With this we can express the Kolmogorov turbulent length scale η as

$$\eta^{-1} = (E/\nu^3)^{\frac{1}{4}} = k_0\sqrt{R} \quad . \tag{9}$$

To have a large inertial subrange $k_0 \ll k \ll \sqrt{R}\,k_0 = \eta^{-1}$, we clearly need large R.

When we use the renormalization group to study the turbulence developed through the mixing (1), we will find below, as in the case $\Gamma_M(k^2)=1$ discussed before,[3] that the $k\to\infty$ limit of the theory is governed by an effective Reynolds number which goes to zero as $k^{-(4-D)/4}$ for D<4. Thus one can use perturbation theory with great accuracy in this wave number range, i.e., $k\eta>1$. Similarly we will find the small wave number range, $k<k_0$ to be determined by an effective Reynolds number which is proportional to $(4-D)^{\frac{1}{2}}$ and is thus small near D=4. [Probably D=3 is close enough.] The intermediate regime $k_0 < k < \eta^{-1} = \sqrt{R}\,k_0$ will be governed by a small coupling as well but will be a bit more complicated in detail than either the large or small k regions.

The plan of this paper is to first provide a review of the renormalization group treatment of stochastic field theory. The role of random mixing forces will be treated more explicitly than before. Next we study in perturbation theory the properties of the renormalization group functions which enter the renormalization group equations for velocity correlation functions. We close with comments and discussion of the results.

II. STOCHASTIC FIELD THEORY AND RENORMALIZATION GROUP

This section will combine a review of salient features of the field theoretic analysis in Ref. 3 with the details needed for a study of the case at hand when an external wave number scale k_0 is present. We will employ a slightly altered renormalization prescription as well.

We want to investigate the stochastic field theory of the velocity field $v_j(\vec{x},t)$ satisfying the Navier-Stokes equation

$$\frac{\partial}{\partial t} v_j(\vec{x},t) + \tfrac{1}{2}(\Delta_{jn}(\nabla)\nabla_\ell + \Delta_{j\ell}(\nabla)\nabla_n)v_n v_\ell$$

$$= \nu_0 \nabla^2 v_j + F_j \tag{10}$$

for an incompressible fluid, $\nabla_j v_j = 0$, in the presence of a random stirring force F_j. We will take F_j to have a gaussian distribution functional

$$P[F_j(\vec{x},t)] = \exp - \frac{1}{2a}\int d^D x dt d^D y d\tau F_j(\vec{x},t)\hat{\Gamma}^{-1}_{j\ell}(\vec{x},t;\vec{y},\tau)F_\ell(\vec{y},\tau) \quad , \tag{11}$$

where a is the stochasticity parameter introduced previously. The mean of F_j in this distribution is zero, and

$$\langle F_j(\vec{x},t)F_\ell(\vec{y},\tau)\rangle_F = \frac{\int dF_n(\vec{u},\rho)F_j(\vec{x},t)F_\ell(\vec{y},\tau)P[F_j]}{\int dF_n(\vec{u},\rho)P[F_j]} \tag{12}$$

$$= a\,\hat{\Gamma}_{j\ell}(\vec{x},t;\vec{y},\tau) \quad . \tag{13}$$

The stochastic nature of v_j is expressed by giving the generating functional for the correlation functions

$$Z[\eta_j,\bar{\eta}_j] = \left\langle \int dv_j\,\delta(\nabla_n v_n)d\bar{v}_\ell\,\delta(\nabla_k\bar{v}_k)e^{-\frac{1}{a}\int d^D x dt[\hat{L}+\bar{\eta}_j\bar{v}_j+\eta_j v_j]}\right\rangle_F \quad , \tag{14}$$

where

$$\hat{L} = \tfrac{1}{2}\bar{v}_j \frac{\overleftrightarrow{\partial}}{\partial t} v_j + \nu_0 \nabla_n\bar{v}_j\nabla_n v_j + F_j\bar{v}_j$$

$$- \tfrac{1}{2}[(\Delta_{jn}(\nabla)\nabla_\ell + \Delta_{j\ell}(\nabla)\nabla_n)\bar{v}_j]\; v_n v_\ell \tag{15}$$

is the Lagrangian density for (10). It involves the "anti-velocity" $\bar{v}_j$ which is necessary because (1) is linear in time derivatives. Using the distribution function $P[F_j]$ we can perform the gaussian integral indicated in (14) to learn

$$Z[\eta_j,\bar{\eta}_j] = \int dv_j\,\delta(\nabla_n v_n)\,d\bar{v}_\ell\,\delta(\nabla_\ell \bar{v}_\ell)\,e^{-\frac{1}{a}\int d^D x\,dt\,[L+\bar{\eta}_j\bar{v}_j+\eta_j v_j]} \quad , \tag{16}$$

with

$$L = \tfrac{1}{2}\bar{v}_j \frac{\overleftrightarrow{\partial}}{\partial t} v_j + \nu_0 \nabla_n \bar{v}_j \nabla_n v_j$$

$$- \tfrac{1}{2}\bar{v}_j \hat{\Gamma}_{j\ell}\bar{v}_\ell - \tfrac{1}{2}[(\Delta_{jn}(\nabla)\nabla_\ell + \Delta_{j\ell}(\nabla)\nabla_n)\bar{v}_j]v_n v_\ell \; . \tag{17}$$

This was the starting point in Ref. 3 and may be found, with appropriate translation of notation, in Ref. 5.

If the distribution functional $P[F_j]$ were not gaussian, the integral over random forces weighted by $P[F_j]$ would be, most likely, impossible to do exactly. However, we are really finding here the characteristic functional of $P[F_j]$ since the integral to do is

$$\int dF_n(\vec{u},\rho)\; P[F_j]\,e^{-\frac{1}{a}\int F_j \bar{v}_j d^D x dt} \quad ,$$

and one may be able to proceed with a cumulant expansion keeping only a few terms.

As explained in the introduction we wish to choose F_j divergenceless and δ-function correlated in time. With a homogeneous, isotropic medium we may then write

$$\hat{\Gamma}_{j\ell}(\vec{x},t;\vec{y},\tau) = \delta(t-\tau)\frac{\gamma_0^{\,2}}{4}\Delta_{j\ell}(\nabla)\tilde{\Gamma}(k_0^{\,2}(\vec{x}-\vec{y})^2) \quad , \tag{18}$$

where $\tilde{\Gamma}(k_0^{\,2}x^2)$ has dimensions $k_0^{\,D}$ so its Fourier transform

$$\Gamma_M(k^2/k_0^2) = \int d^D x\; e^{-i\,\vec{k}\cdot\vec{x}}\;\tilde{\Gamma}(k_0^{\,2}x^2) \tag{19}$$

is dimensionless. Novikov[5] has shown that the net energy dissipation, $\mathcal{E}$, due to viscous forces is

$$\mathcal{E} = \frac{(D-1)}{2} \frac{\gamma_0^2}{4} \tilde{\Gamma}(0) = \frac{D-1}{2} \int \frac{d^D p}{(2\pi)^D} \Gamma_M(p^2/k_0^2) \frac{\gamma_0^2}{4} , \tag{20}$$

$$= \gamma_0^2 k_0^D \frac{D-1}{8} \int \frac{d^D p}{(2\pi)^D} \Gamma_M(p^2) . \tag{21}$$

Clearly to have a realistic model of turbulence we must have finite $\mathcal{E}$. So $\tilde{\Gamma}(0)$ should be non-zero, and $\Gamma_M(k^2)$ must fall rapidly enough for (21) to converge. Furthermore we would like $\tilde{\Gamma}(k_0^2 x^2)$ to peak around the lengths k_0^{-1} which are important in the mixing force. If we develop the turbulence by passing fluid through a screen with grid spacing L,[4] the $\tilde{\Gamma}(k_0^2 x^2)$ should peak near $x = L$ or $k_0 x = x/L = 1$. Many such functions are easy to imagine, and very little of what we say below depends on the detailed form of $\tilde{\Gamma}(k_0^2 x^2)$. It will, however, be important, from a mathematical point of view, whether $\Gamma_M(k^2/k_0^2) = 0$ or not. Equivalently whether

$$\int d^D x \, \tilde{\Gamma}(k_0^2 x^2) = 0 \tag{22}$$

or not. A $\tilde{\Gamma}(k_0^2 x^2)$ which begins from some non-zero value at $x=0$ (so $\mathcal{E} \neq 0$) and peaks near $k_0 x \simeq 1$ and then smoothly and rapidly falls to zero for $x > k_0^{-1} = L$ will not give zero for (22). It would seem then to cover most cases of physical interest. For completeness, however, we will also discuss the case $\Gamma_M(k^2=0) = 0$ below.

A last comment about the forcing function is in order. With no forcing or mixing of the fluid the only stationary state of motion is clearly $v_j=0$, since the kinetic energy will certainly be dissipated by the viscous forces. Driving the fluid by a gaussian random function is a more or less realistic representation of the manner in which turbulent motion is realized; it is, however, a useful device to simulate the effect of random boundary conditions or initial conditions which may actually be responsible for generating the turbulent motion. The very interesting situation where the fluid is stirred by a random force which is then turned off allows us to study the decay of the turbulent motion. In that case we would choose

$$\tilde{\Gamma}_{j\ell}(\vec{x},t;\vec{y},\tau) = \frac{\gamma_0^2}{4}\Delta_{j\ell}(\nabla)\tilde{\Gamma}(k_0^2(\vec{x}-\vec{y})^2)\,\theta(-\frac{(t+\tau)}{2})\,\delta(t-\tau) \tag{23}$$

for stirring turned on long ago and turned off at t=0. The study of the turbulent motion from this mixing at large positive times will be the subject of the next paper of this series.

Now we return to the Lagrangian density (17) and rescale the v_j and $\bar{v}_j$ by

$$v_j(\vec{x},t) = \frac{\gamma_0}{2}\chi_{0_j}(\vec{x},t) \quad , \tag{24}$$

and

$$\bar{v}_j(x,t) = 2\gamma_0^{-1}\bar{\chi}_{0_j}(\vec{x},t) \tag{25}$$

which makes it quite clear that γ_0 is the parameter which sets the scale of the non-linearity since

$$\begin{aligned} L[\chi_{0j},\bar{\chi}_{0j}] &= \tfrac{1}{2}\,\bar{\chi}_{0j}\overleftrightarrow{\partial}_t\chi_{0j} + \nu_0\nabla_n\bar{\chi}_{0j}\nabla_n\chi_{0j} - \tfrac{1}{2}\,\bar{\chi}_{0j}\Gamma\bar{\chi}_{0j} \\ &- \frac{\gamma_0}{4}\,[\Delta_{jn}(\nabla)\nabla_\ell + \Delta_{j\ell}(\nabla)\nabla_n)\bar{\chi}_{0j}]\chi_{0n}\chi_{0\ell} \quad . \end{aligned} \tag{26}$$

In Ref. 3 we gave the rules for constructing the correlation functions $G^{(n,m)}$ of n χ_{0j} fields and m $\bar{\chi}_{0j}$ fields as a power series in γ_0. In the present case the rules are modified only by a change in the unperturbed correlation function $\langle\chi_{0j}\chi_{0\ell}\rangle$. It is called $D^0_{j\ell}(\vec{k},\omega)$ in Ref. 3 and now reads

$$D^{0}_{j\ell}(\vec{k},\omega) = \Delta_{j\ell}(\vec{k}) \frac{-\Gamma_M(k^2/k_0{}^2)}{(i\omega+\nu_0\vec{k}^2-\varepsilon)(i\omega-\nu_0\vec{k}^2-\varepsilon)} \quad . \tag{27}$$

We wish to use the renormalization group to study properties of the theory defined by (26) beyond the confines of perturbation theory. This tool enteres the scene when we study the consequences of replacing the quantitites χ_{0j}, $\bar{\chi}_{0j}$, ν_0, and γ_0 by the rescaled objects

$$\chi_j(\vec{x},t) = Z^{-\frac{1}{2}}\chi_{0j}(\vec{x},t) \quad , \tag{28}$$

$$\bar{\chi}_j(\vec{x},t) = \bar{Z}^{-\frac{1}{2}}\bar{\chi}_{0j}(\vec{x},t) \quad , \tag{29}$$

$$\nu = Z_\nu \, \nu_0 \quad , \tag{30}$$

and

$$\gamma = Z_\gamma \, \gamma_0 \quad , \tag{31}$$

by giving the value of certain of the $G^{(n,m)}$ at an arbitrary point in k,ω space. The requirement that the physical consequnces of the theory be independent of this arbitrary normalization point gives constraints on all correlation functions known as the renormalization group equations.

Our renormalization procedure consists in giving values for certain derivatives of the renormalized correlation functions

$$G_j^{(1,1)}(k^2,\omega) = \int d^D x dt e^{i(\vec{k}\cdot\vec{x}-\omega t)} \langle T(\chi_j(\vec{x},t)\bar{\chi}_\ell(0,0))\rangle \tag{32}$$

$$= \Delta_{j\ell}(k) \; G_R^{(1,1)}(\vec{k}^2,\omega) \tag{33}$$

and

$$G_{j\ell}^{(2,0)}(\vec{k},\omega) = \int d^D x dt e^{i(\vec{k}\cdot\vec{x}-\omega t)} \langle T(\chi_j(\vec{x},t)\chi_\ell(0,0))\rangle \tag{34}$$

$$= \Delta_{j\ell}(k) \; G_R^{(2,0)}(\vec{k}^2,\omega) \tag{35}$$

and of the fusion vertex $\Gamma_{jn\ell}$ corresponding to the non-linearity in (26).

These conditions are

$$\frac{\partial}{\partial\omega} G_R^{(1,1)} (\vec{k}^2, \omega)^{-1} \Bigg|_{\substack{\omega = i\omega_N \\ \vec{k}^2=0}} = -i \tag{36}$$

$$\frac{\partial}{\partial\vec{k}^2} G_R^{(1,1)} (\vec{k}^2, \omega)^{-1} \Bigg|_{\substack{\omega = i\omega_N \\ \vec{k}^2=0}} = \nu \tag{37}$$

as suggested by the unperturbed value

$$G_0^{(1,1)} (\vec{k}^2, \omega)^{-1} = -i\omega + \nu_0 \vec{k}^2 \quad . \tag{38}$$

Also we choose

$$\frac{\partial}{\partial\omega} G_R^{(2,0)} (\vec{k}^2, \omega)^{-1} \Bigg|_{\substack{\vec{k}^2=0 \\ \omega = i\omega_N}} = 2i\omega_N \tag{39}$$

and

$$\frac{k_j \Gamma_{jn\ell}(\vec{k}, \omega; \vec{q}_1, \omega_1, \vec{q}_2, \omega_2)\, \delta_{n\ell}}{k^2} \Bigg|_{\substack{\vec{k} = \vec{q}_1 = \vec{q}_2 = 0 \\ \omega = 2\omega_1 = 2\omega_2 = i\omega_N}} = \frac{-i\gamma}{(2\pi)^{\frac{D+1}{2}}} \quad , \tag{40}$$

again suggested by the unperturbed values as given in (27) and

$$\Gamma^0_{jn\ell} = \frac{-i\gamma_0}{2(2\pi)^{\frac{D+1}{2}}} (\delta_{jn} k_\ell + \delta_{j\ell} k_n) \quad . \tag{41}$$

There is only one change here from Ref. 3. We now take the normalization point in $\vec{k}, \omega$ space to have all $\vec{k}_j = 0$ and ω_j proportional to $i\omega_N$, ω_N real. This guarantees that all of the rescaling factors for $\chi_j, \bar{\chi}_j, \nu$

and γ are real. The factors Z and $\bar{Z}$ are still equal one.[3]

The dimensional analysis performed before holds now, but we find an additional dimensionless parameter because of the presence of k_0. So we define

$$\sigma = \omega_N/k_0^{\,2}\nu = k_N^{\,2}/k_0^{\,2} \quad , \tag{42}$$

introducing $k_N^{\,2} = \omega_N/\nu$. The other dimensionless parameter is the Reynolds number based on the scale k_N^{-1}. It is the one in which we are doing (renormalized) perturbation theory and is

$$g = \frac{\gamma}{\nu^{3/2}}\, k_N^{(D-4)/2} \tag{43}$$

$$= R\, \sigma^{(D-4)/4} \quad . \tag{44}$$

Now the renormalization group equations tell us how g, ν, σ and $\phi^{(n,m)} = (\frac{\gamma}{2})^{n-m} G_R^{(n,m)}$, the n v_j and m $\bar{v}_j$ correlation function, must simultaneously vary so variations in $k_N^{\,2}$ don't affect the physics. It now takes the form:

$$\left[k_N^{\,2} \frac{\partial}{\partial k_N^{\,2}} + \frac{A(g,\sigma)}{1-B(g,\sigma)} \frac{\partial}{\partial g} + \frac{B(g,\sigma)}{1-B(g,\sigma)}\, \nu \frac{\partial}{\partial \nu} + \sigma \frac{\sigma}{\partial \sigma} + \right.$$

$$\left. + C_{n,m}(g,\sigma)\right] \phi^{(n,m)} (\vec{k}_i, \omega_i, g, \nu, \sigma, k_N^{\,2}) = 0 \tag{45}$$

where

$$A(g,\sigma) = \omega_N \frac{\partial}{\partial \omega N}\, g \Bigg|_{\gamma_0, \nu_0, k_0 \text{ fixed}} \quad , \tag{46}$$

$$B(g,\sigma) = \frac{\omega_N}{\nu} \frac{\partial}{\partial \omega_N}\, \nu \Bigg|_{\gamma_0, \nu_0, k_0 \text{ fixed}} \quad , \tag{47}$$

and

$$C_{n,m}(g,\sigma) = (m-n)\left[1 - \frac{D}{4} + \frac{(D+2)}{4} B(g,\sigma) + \frac{A(g,\sigma)}{g}\right] \quad . \tag{48}$$

The solution to this equation is given in terms of the effective Reynolds number $\tilde{g}(u)$, the effective viscosity $\tilde{\nu}(u)$, and the effective scale ratio $\tilde{\sigma}(u)$ satisfying

$$\frac{d\tilde{g}(u)}{du} = \frac{-A(\tilde{g}(u),\tilde{\sigma}(u))}{1-B(\tilde{g}(u),\tilde{\sigma}(u))} \quad , \tag{49}$$

$$\frac{d\tilde{\sigma}(u)}{\tilde{\nu}(u)du} \quad \frac{-1}{1-B(\tilde{g}(u),\tilde{\sigma}(u))} \quad , \tag{50}$$

$$\frac{d\tilde{g}(u)}{du} = -\tilde{\sigma}(u) \quad , \tag{51}$$

with the boundary conditions $\tilde{g}(0) = g$, $\tilde{\nu}(0) = \nu$, $\tilde{\sigma}(0) = \sigma$. Clearly $\tilde{\sigma}(u) = \sigma e^{-u}$. Then $\Phi^{(n,m)}$ satisfies

$$\Phi^{(n,m)}(\sqrt{\xi}\vec{k}_i,\omega_i,g,\nu,\sigma,k_N^2) =$$

$$\Phi^{(n,m)}(\vec{k}_i,\omega_i,\tilde{g}(-\log\,\xi),\tilde{\nu}(-\log\xi),\xi\sigma,k_N^2) \times$$
$$\exp\int_{-\log\xi}^{0} du\ \gamma_{n,m}\left[\tilde{g}(u),\tilde{\sigma}(u)\right] \quad , \tag{52}$$

with

$$\gamma_{n,m}(g,\sigma) = \frac{(m-n)(2-D)+2(1-n)}{4} + \frac{(m-n)}{1-B(g,\sigma)}\left[1+\frac{B(g,\sigma)}{2}+\frac{A(g,\sigma)}{g}\right] \quad . \tag{53}$$

This formula for $\Phi^{(n,m)}$ allows us to explore the $\vec{k}_i$ dependence of the velocity correlation functions for turbulence. In any regime of k-space where $\tilde{g}(-\log\xi)$ is small, we may determine everything by perturbation theory for $\Phi^{(n,m)}(\vec{k}_i,\omega_i,\tilde{g},\tilde{\nu},\xi\sigma,k_N^2)$. This will be true for large k. For small k we found in Ref. 3 that A(g) had a zero at $g_1 \alpha \sqrt{4-D}$ with $dA/dg|_{g_1} > 0$, so that for $D \tilde{=} 4$ we could determine the small k behavior of the theory as well. Next we explore the behavior of turbulence mixed by our cutoff

$\Gamma(k^2/k_0^2)$ by examining A and B in perturbation theory. It is important to emphasize that perturbative knowledge of A and B gives us non-perturbative constraints on $\Phi^{(n,m)}$ via (45).

IV. RENORMALIZATION GROUP FUNCTIONS AND EFFECTIVE COUPLINGS

To determine the renormalization group functions we turn, as usual, to perturbation theory in g . The information we require for $B(g,\sigma)$ comes from the normalization condition (37) which determines Z_ν since

$$B(g,\sigma) = \omega_N \frac{\partial}{\partial\omega_N} \log Z_\nu \Big|_{\gamma_0,\nu_0,k_0 \text{ fixed}} \quad . \tag{54}$$

From the graphs in Fig. 1 we find

$$B(g,\sigma) = \frac{-g^2}{D+2}\,\frac{\sigma^{3-D/2}}{32D}\int \frac{d^Dq}{(2\pi)^D}\,\frac{\Gamma_M(q^2)}{q^2}\,\frac{\left[q^2(D^2-D+2)+\frac{\sigma}{2}(D^2-D-2)\right]}{(q^2+\frac{\sigma}{2})^3} \tag{55}$$

$$= \frac{-g^2}{D+2}\,\frac{1}{32D}\int \frac{d^Dp}{(2\pi)^D}\,\frac{\Gamma_M(\sigma p^2)}{p^2(p^2+\frac{1}{2})^3}\left[p^2(D^2-D+2)+\tfrac{1}{2}(D^2-D-2)\right] \quad , \tag{56}$$

$$= -g^2F(\sigma)/D+2 \quad . \tag{57}$$

The graphs in Fig. 2 give a net contribution zero with our normalization condition (40). So for $A(g,\sigma)$ we have

$$A(g,\sigma) = -\frac{\varepsilon}{4}\,g + \frac{F(\sigma)}{4}\,g^3 \;, \quad \varepsilon = 4-D \quad . \tag{58}$$

The behavior of the velocity correlation functions $\Phi^{(n,m)}$ is determined by $\bar{g}(u)$ and $\bar{\nu}(u)$ as discussed above. The study of $\bar{g}(u)$ is most important; it satisfies

$$\frac{d\tilde{g}(u)}{du} = \frac{\tilde{g}(u)}{4}\,\frac{(\varepsilon-g(u)^2F(\tilde{\sigma}))}{1+\frac{\tilde{g}(u)^2F(\tilde{\sigma})}{D+2}} \quad . \tag{59}$$

We study the large wave number limit of the $\Phi^{(n,m)}$ by examining (59) for $u = -\log \xi$ as $\xi\to\infty$, so $u\to -\infty$; $\tilde{\sigma}(u) = \sigma e^{-u}\to\infty$. From (55) we see that

$$F(\sigma) \underset{\sigma\to\infty}{\sim} \frac{\sigma^{1-D/2}}{8D}\int\frac{d^Dq}{(2\pi)^D}\,\frac{\Gamma_M(q^2)}{q^2}\,(D^2-D-2) \quad , \tag{60}$$

so for D>2, $F(\sigma)$ vanishes rapidly for large σ . The equation for $\tilde{g}(u)$ becomes

$$\frac{d\tilde{g}(u)}{du} = \frac{\varepsilon}{4}\tilde{g}(u) \tag{61}$$

and

$$\tilde{g}(-\log \xi) = g\ \xi^{-\varepsilon/4} \tag{62}$$

for large ξ .

This is just what we found in Ref. 3. It means that for large wave number the $\Phi^{(n,m)}$ may be accurately evaluated in perturbation theory in $\tilde{g}(-\log \xi)$. Since $\tilde{g}$ measures the "size" of the non-linearity relative to the linear terms in the Navier-Stokes equations, this result makes good physical sense. For large k^2 we expect the viscous term to dominate the non-linear inertial term. Indeed, that is what happens for $D<4(\varepsilon>0)$. This effect is also apparent in the effective viscosity what behaves as

$$\tilde{\nu}(-\log \xi) = \xi\ \nu \tag{63}$$

for large ξ.

For the behavior of the theory at small wave number, we need some information on $\Gamma_M(k^2)$ for small k^2; this is clear from (56). Let's suppose that $\Gamma_M(\sigma\ p^2)$ is such that for $\sigma\to 0$, $F(\tilde{\sigma}) \sim c_0\ e^{-Nu}$. Now our equation for $\tilde{g}(u)$ as $u\to+\infty(\xi\to 0)$, is

$$\frac{d\tilde{g}(u)}{du} = \frac{\tilde{g}(u)}{4} \frac{(\varepsilon - \tilde{g}(u)^2 c_0\ e^{-Nu})}{1 + \dfrac{\tilde{g}(u)^2 c_0\ e^{-Nu}}{D+2}} \quad . \tag{64}$$

Introduce $f(u) = e^{-Nu}\tilde{g}(u)^2$, then

$$\frac{df(u)}{du} = \frac{-f(u)}{1 + \dfrac{f(u)\ c_0}{D+2}} \left[(N - \frac{\varepsilon}{2}) + c_0 (N+\tfrac{1}{2}) f(u) \right] \quad . \tag{65}$$

If $N > \varepsilon/2$, then as $u\to\infty$,

$$f(u) \sim e^{-(N-\varepsilon/2)u} \quad , \tag{66}$$

while

$$\tilde{g}(u) \underset{u\to\infty}{\sim} e^{\varepsilon u/4} \quad , \tag{67}$$

so it becomes infinite. If $N < \varepsilon/2$, then $f(u)$ approaches the finite value

$$f(u) \underset{u\to\infty}{\sim} \frac{(\varepsilon/2-N)}{c_0 (N+\frac{1}{2})} \tag{68}$$

while

$$\tilde{g}(u)^2 \underset{u\to\infty}{\sim} e^{Nu} \left[\frac{\varepsilon - 2N}{c_0 (2N+1)} \right] \tag{69}$$

which is again infinite for $N > 0$.

At this state the importance of the properties of $\Gamma_M(k^2)$ for small k^2 have emerged. From a physical point of view it seems unlikely that $\Gamma_M(k^2=0)$ should be anything but a finite number. Since

$$\Gamma_M(k^2=0) = \int d^D x\ \tilde{\Gamma}(x^2) \quad , \tag{70}$$

Reply to the Referee on "The Behavior of Homogeneous Turbulence Mixed at Long Wavelengths"

I appreciate the long and careful review given by the referee of my paper. I will try to address the two important points he raises: (1) a question about the behavior of the effective coupling as one varies the bare Reynolds number, and (2) the phrasing on p. 17-19 about $\Gamma_M(0)$.

(1) The renormalized Reynolds number, g, is an infinite series in g_o, the bare or unrenormalized Reynolds number. If the function A(g) has a zero with positive slope at $g = g_1$ and a zero with negative slope at $g = 0$; e.g.

$$A(g) = -\frac{\varepsilon}{4} g + a g^3 \quad , \quad a > 0 \; ,$$

as in the theory of turbulence, then one can solve for the $g(g_o)$ relation by using the boundary condition that when $g_o \to 0$, $g \to 0$. The relation, good at the same level as A(g), is

$$g^2 = \frac{g_o^{\,2}}{1 + g_o^{\,2}/g_1^{\,2}} \quad .$$

From this one sees that as the bare Reynolds number g_o ranges over $0 \leq g_o < \infty$, g ranges from zero only up to g_1. The effective coupling $\tilde{g}$ reflects this behavior. As $k \to \infty$, if $g \to 0$ for any g_o, it goes to zero for every g_o.

I believe there is an important physical point to be made, and perhaps that is what the referee is driving at. $k \to \infty$ literally means leaving the inertial range and moving into the deep dissipation range where $k \gg \eta^{-1} = (\varepsilon/\nu^3)^{1/4}$. In that range, whatever the bare Reynolds number for $D < 4$, $\nu\nabla^2 v_j$ will dominate the inertial term. Perhaps I am wrong, but I don't see a possible disagreement with that. The issue, then, is the behavior in the intermediate regime where k is large compared to 0 or k_o, but still outside the deep dissipation regime. Here the interpolating formulae derived in Section V of this paper are the tool to explore this region.

As to the behavior of $\tilde{g}$ changing as R_o changes, I cannot agree. I recommend the referee explore the field theory of a scalar field with $\lambda_o \phi^4$ coupling in D dimensions. The nature of perturbation theory in λ_o changes at D = 4, regardless of the size of λ_o. For D < 4 two phases of the theory are possible. One is connected to perturbation theory and has a dissipation region where as $k \to \infty$, $\lambda_{effective} \to 0$. The other has $\lambda_{effective} \to \infty$ as $k \to \infty$. For $D \geq 4$ only one phase exists. The presence of two phases is not dependent on the size of λ_o. The turbulence problem is much the same. Here, however, we are fortunate in having a physical boundary condition to choose the appropriate phase for D < 4. That boundary condition is the existence of a dissipation region where $\nu\nabla^2 v_j$ dominates $\vec{v}\,\nabla v_j$ and the effective Reynolds number goes to zero.

(2) On rereading the paragraph beginning on the bottom of p. 18 I can see how it should be rewritten for clarity. I enclose an altered paragraph to address that.

I think this should make it clear that my preference for $\Gamma_M(0) \neq 0$ has a sound physical basis.

One last comment which pertains to the referee's statement about E(k) decreasing as $\Gamma_M(k)$. For very large k, i.e. in the deep dissipation region that is what physically one would expect some transport by $\vec{v}\cdot\nabla v_j$ has become unimportant. For large k but still less than η^{-1} there is a combination of effects consisting of energy transport by $\vec{v}\cdot\nabla v_j$ and decreased input due to the fall off of Γ_M. It seems to me possible, though not yet demonstrated that a balance yielding a Kolmogorov spectrum could arise, though I do not expect it in a neat analytic sense.

Substitution for paragraph at bottom of page 18:

At this stage an aside is in order. By looking at the time dependence of the decay of homogeneous, isotropic turbulence in the final stages of decay one can learn directly about $\Gamma_M(k^2 = 0)$. There are two competing hypotheses about the behavior of $\Gamma_M(0)$. One is given by Batchelor,[4] Section 5.4, where he argues that the analyticity of the velocity correlation function at $k = 0$ requires

$$\Gamma_M(k^2) \backsim k^2 \tag{75}$$

near $k^2 = 0$. This has been criticized in detail by Saffmann[9] who argues instead that the analyticity assumption is more properly made about the vorticity correlation function. Then one has $\Gamma_M(0)$ finite and, furthermore, an invariant of the motion. An additional argument in favor of Saffman's conjecture is that $\Gamma_M(0) \neq 0$ would imply, for long times into the decay period when the degrees of freedom of the fluid had time to come to equilibrium after whatever mixing had occurred, that the energy spectrum $E(k)$ behaves as k^{D-1} which one expects from equipartition. It is important to note that there is a difference between $E(k, t)$ in non-stationary turbulence and $E(k)$ in the stationary case. It is $E(k, t)$ which for long times after the mixing behaves at $k \to 0$ as k^{D-1} when $\Gamma_M(0) \neq 0$. $E(k)$ has an additional factor of k^{-2} and behaves as $E(k) \backsim k^{D-3}$ when $\Gamma_M(0) \neq 0$. For generality, however, it is easy enough to consider a behavior like (75). Then $N = 1$ at $D = 3$, and $\tilde{g}(u) \to \infty$ as $u \to \infty$ or $\xi \to 0$.

Add to references

[9] P.G. Saffman, J. Fluid. Mech. 27, 581 (1967).

In first paragraph on p. 21 change first sentence to:

For the circumstances mentioned above $\Gamma_M(k^2) \propto k^2$ for small k, N = 1, and we see that $\tilde{G}(u)$ is zero at both ends of the wave number spectrum for D = 3.

it is clear that $\tilde{\Gamma}(x^2)$, the random force correlation function in ordinary space, must oscillate in some fashion to make $\Gamma_M(k^2=0)=0$. Furthermore, since $E=\tilde{\Gamma}(0)\neq 0$, and we expect $\tilde{\Gamma}(x^2)\to 0$ as $x\to\infty$, singular behavior of $\Gamma_M(k^2)$ at $k^2=0$ also appears unlikely.

We can henceforth safely assume $\Gamma_M(0)$ is some finite number which we choose by convention to be unity. From (56) we see then that

$$F(\sigma=0) = \frac{1}{32D}\int \frac{d^Dp}{(2\pi)^D}\,\frac{p^2(D^2-D+2)+\frac{1}{2}(D^2-D-2)}{p^2(p^2+\frac{1}{2})^3} \quad , \tag{71}$$

which corresponds to the case N=0 above. The effective Reynolds number for small wave number approaches

$$\bar{g}(-\log\,\xi) \underset{\xi\to 0}{\sim} \sqrt{\frac{\varepsilon}{c_0}} \tag{72}$$

and

$$c_0 = F(\sigma=0) \quad . \tag{73}$$

This is just the situation encountered in Ref. 3 as σ no longer enters. Using the idea that $\varepsilon=4-D$ is small, we can then evaluate c_0 at $\varepsilon=0$ and expand appropriate quantities which are power series in $\bar{g}$ in powers of ε. In this instance we know[3] that

$$\frac{\bar{g}(-\log\,\xi)}{4\pi} \underset{\xi\to 0}{\sim} \sqrt{\frac{8\varepsilon}{3}} \quad . \tag{74}$$

At this stage an aside is in order. By looking at the time dependence of the decay of homogeneous, isotropic turbulence in the final stages of decay Batchelor (Ref. 4, Sec. 5.4) presents evidence that $\Gamma_M(k^2=0)$ is zero and, indeed, behaves as

$$\Gamma_M(k^2) = k^4 h_M(k^2) \tag{75}$$

with $h_M(0)$ finite. Monin and Yaglom (Ref. 1, Secs 15.3-15.5) review and criticize this result but suggest that the behavior of $\Gamma_M(k^2)$ may be as k^2 for $k^2 \to 0$. In these cases N, above, is not zero and the effective Reynolds number g(u) becomes infinite for low wave numbers. Examination of $F(\sigma)$ for these cases shows that when $\Gamma_M \propto k^4$, then $N=1+\varepsilon/2$ and when $\Gamma_M(k^2) \propto k^2$, then $N=\frac{1}{2}$ in D=3. In each case $\tilde{g}(u) \to \infty$ as $u \to \infty$ or $\xi \to 0$.

This circumstance, if correct, can be treated by noting that the expansion parameter is not g^2 only, but is g^2 times some function of σ and the interplay between $\tilde{g}$ and $\tilde{\sigma} = \sigma e^{-u}$ is essential. Suppose then we identify the dimensionless expansion parameter in the functions $A(g,\sigma)$ or $B(g,\sigma)$ or $\phi^{(n,m)}$ to be

$$G = g\Lambda(\sigma) \tag{76}$$

where we will choose $\Lambda(\sigma)$ to make the behavior in $\tilde{G}(u)$ as smooth as possible. Now

$$\frac{d\tilde{G}}{du} = \frac{d\tilde{g}}{du}\Lambda(\tilde{\sigma}) + \tilde{g}\frac{d}{du}\Lambda(\tilde{\sigma}) \tag{77}$$

$$= \frac{-A(\tilde{g},\tilde{\sigma})}{1-B(\tilde{g},\tilde{\sigma})}\Lambda(\tilde{\sigma}) + \tilde{G}\frac{d}{du}\log\Lambda(\sigma) \quad . \tag{78}$$

We have to lowest order in g^2

$$A(g,\sigma) = -\frac{\varepsilon}{4}g + \frac{F(\sigma)}{4}g^3$$

so

$$\frac{d\tilde{G}(u)}{du} = \frac{\frac{\varepsilon}{4}\tilde{G} - \frac{\tilde{G}^3 F(\tilde{\sigma})}{4\Lambda(\tilde{\sigma})^2}}{1 + \frac{\tilde{G}^2 F(\tilde{\sigma})}{(D+2)\Lambda(\tilde{\sigma})^2}} + \tilde{G}(u)\frac{d}{du}\log\Lambda(\tilde{\sigma}) \quad , \tag{79}$$

so it seems we should choose

$$\Lambda(\sigma) = F(\sigma)^{\frac{1}{2}} \tag{80}$$

and

$$\frac{d\tilde{G}}{du} = \tilde{G}\left(\frac{\varepsilon}{4} + \frac{1}{2}\frac{d}{du}\log F(\tilde{\sigma})\right) - \frac{\tilde{G}^3}{4}\left(1 + \frac{\varepsilon}{D+2}\right) \tag{81}$$

keeping consistent powers of g (or G).

As $u \to -\infty$ or $\xi \to +\infty$, the large wave number limit of the theory, we have

$$F(\sigma) \sim \sigma^{1-D/2}$$

which results in

$$\tilde{G}(u) \underset{u\to-\infty}{\sim} e^{u/2}$$

which means our modified effective coupling constant (expansion parameter) is small in the large k^2 limit. For small $k^2 (u \to +\infty)$ we again take $F(\tilde{\sigma}) \sim e^{-Nu}$. Then we can solve for $\tilde{G}(u)$ from (81), and find

$$\tilde{G}(u)^2 = \frac{\tilde{G}(0)^2}{1 + \frac{\tilde{G}(0)^2}{2}\frac{(1+\varepsilon/D+2)}{\frac{\varepsilon}{2}-N}\left(e^{(\frac{\varepsilon}{2}-N)u} - 1\right)} \quad . \tag{82}$$

So when $\varepsilon/2 > N$, we have

$$\tilde{G}(u)^2 \underset{u\to+\infty}{\sim} \frac{\varepsilon - 2N}{(1+\varepsilon/D+2)}$$

which, as expected, is $O(\varepsilon)$ for $N=0$. When $\varepsilon=2N$, $\tilde{G}(u)^2$ goes to zero as

$$\tilde{G}(u)^2 \underset{u\to+\infty}{\sim} \frac{2}{(1+\varepsilon/D+2)u} \quad ,$$

while, when $N>\varepsilon/2$, $\tilde{G}(u)$ again goes to zero, but now

$$\tilde{G}(u)^2 \underset{u\to\infty}{\sim} e^{-(N-\frac{\varepsilon}{2})u} .$$

For the circumstances mentioned above: $\Gamma_M(k^2) \propto k^4$ for small k, $N=1+\frac{\varepsilon}{2}$ and we see that $\tilde{G}(u)$ is zero at both ends of the wave number spectrum. For intermediate wave numbers a treatment similar to the one given in Sec. V below is needed.

Now we return to the case $\Gamma_M(k^2=0) = 1$ whose physical motivation we have discussed. In this situation the effective Reynolds number runs between zero at large k and $O(\sqrt{\varepsilon})$ at small k. It would suggest then that a perturbation series in $\tilde{g}$ to determine the velocity correlation functions $\Phi^{(n,m)}$ would be appropriate.

We really have to a bit more careful here if we wish to be numerically accurate for large Reynolds number. The renormalization group as presently formulated probes variations of $\Phi^{(n,m)}$ on the scale of $k_N{}^2$. The ratio $\sigma = k_N{}^2/k_0{}^2$ is

$$\frac{k_N{}^2}{k_0{}^2} = \left(\frac{R}{g}\right)^{4/4-D} \tag{83}$$

and will be very large when R is large. Indeed, at $D = 3$ we have $k_N \approx R^2 k_0$. To make an accurate determination of the variations of the

$\phi^{(n,m)}$ on a wave number scale $\simeq k_0$ or in the inertial range $k_0 \ll k \ll \sqrt{R}\, k_0$ we will have to examine a resummation of the perturbation theory in g. This is accomplished by the techniques of Refs. 6 and 7 developed in high energy physics.

To see these remarks in context let us look at two forms for $G^{(1,1)}$ calculated to order g^2 from the graphs in Fig. 1. First we write the answer with all wave numbers scaled to k_N; $k^2 = \kappa^2 k_N^2$:

$$k_N^2 G^{(1,1)}(k^2,\omega,\nu,g,\sigma) = \frac{1}{\frac{-i\omega}{\nu k_N^2} + \kappa^2}\left[1 - \frac{g^2}{2(2\pi)^D}\frac{1}{\frac{-i\omega}{\nu k_N^2} + \kappa^2} \times \right.$$

$$\left. \times \int \frac{d^D p\,\Gamma_M(\sigma p^2)\left\{\kappa^2 - \frac{(\vec{\kappa}\cdot\vec{p})^2}{p^2} - \frac{2\left(p^2 - \frac{(\vec{\kappa}\cdot\vec{p})^2}{\kappa^2}\right)}{D-1}\frac{\vec{\kappa}\cdot(\vec{p}-\vec{\kappa})}{(\vec{p}-\vec{\kappa})^2}\right\}}{p^2\left[\frac{-i\omega}{\nu k_N^2} + p^2 + (\vec{p}-\vec{\kappa})^2\right]}\right] . \tag{84}$$

In this form it is clear that for variations of k on the scale of k_N, the correction term is small both because we are to use $\tilde{g}$ and $\tilde{\nu}$ and because σ in Γ_M is large.

Next rewrite the same expression by scaling the wave numbers by k_0, $k^2 = k_0^2 \ell^2$:

$$k_0^2 G^{(1,1)}(k^2,\omega,\nu,g,\sigma) = \frac{1}{\frac{-i\omega}{\nu k_0^2} + \ell^2}\left[1 - \frac{R^2}{2(2\pi)^D}\frac{1}{\frac{-i\omega}{\nu k_0^2} + \ell^2}\right.$$

$$\left. \times \int \frac{d^D p\,\Gamma_M(p^2)}{p^2\left[\frac{-i\omega}{\nu k_0^2} + p^2 + (\vec{p}-\vec{\ell})^2\right]}\left\{\ell^2 - \frac{(\vec{\ell}\cdot\vec{p})^2}{p^2} - \frac{2\left(p^2 - \frac{(\vec{\ell}\cdot\vec{p})^2}{\ell^2}\right)}{D-1}\frac{\vec{\ell}\cdot(\vec{p}-\vec{\ell})}{(\vec{p}-\vec{\ell})^2}\right\}\right] . \tag{85}$$

This form indicates that for large R, the correction term would swamp the first order term for wave numbers on the scale of k_0. When using the renormalization group we must, of course, replace R by

$$\tilde{R}(-\log k/k_0) = \tilde{g}(-\log k/k_0)\sigma^{(4-D)/4}\left(\frac{k}{k_0}\right)^{(4-D)/4} \quad , \tag{86}$$

which does eventually cut off the correction term for k/k_0 small. Until our resummation is performed, however, one must be rather wary of accepting perturbation theory as numerically accurate for $k \tilde{=} k_0$. Let us now turn to that.

V. USING THE RENORMALIZATION GROUP TO EVALUATE THE VELOCITY CORRELATION FUNCTION

The full content of the renormalization group constraints on $\Phi^{(2,0)}_{j\ell}$ are contained in (52) and have in essence been given in Ref. 3 and in the previous section. In this section we want to show how to derive an expression for $\Phi^{(2,0)}_{j\ell}$ which has the correct large and small k behavior as determined above and, in addition, allows us to interpolate between these limits. Knowing that $\tilde{g}(-\log \xi)$ always lies between 0 and $O(\sqrt{\varepsilon})$ allows us to make a perturbation expansion in (52), but generally does not yield a compact and manageable form. So we proceed differently.[6,7]

We will calculate $G^{(2,0)}(k^2,\omega,\gamma_0,\nu_0,k_0)$, as defined in (35), and recover $\Phi^{(2,0)}_{j\ell}$ by

$$\Phi^{(2,0)}_{j\ell}(k^2,\omega,\gamma_0,\nu_0,k_0) = \frac{\gamma_0^2}{4}\Delta_{j\ell}(k)\; G^{(2,0)}(k^2,\omega,\gamma_0,\nu_0,k_0) \quad . \tag{87}$$

We are concentrating on the unrenormalized, but complete—including all fluctuations—correlation function, as that is the physically measurable quantity.

Our goal is to determine the dimensionless factor Z given as

$$G^{(2,0)}(\omega,k^2,\gamma_0,\nu_0,k_0) = \frac{\Gamma_M(k^2/k_0^{\,2})}{\omega^2+\nu_0^{\,2}k^4}\, Z(\bar{g}_0,\bar{x}_0,\bar{\sigma}_0) \tag{88}$$

where

$$\bar{g}_0 = \frac{\gamma_0}{\nu_0^{\,3/2}}\,(k^2)^{(D-4)/4} \quad , \tag{89}$$

$$\bar{x}_0 = \frac{\nu_0 k^2}{\omega} \quad , \tag{90}$$

and

$$\bar{\sigma}_0 = k^2/k_0^{\,2} \quad . \tag{91}$$

Since the coefficient of Z in (88) is precisely the value of $G^{(2,0)}$ for $\gamma_0 = 0$, we know

$$Z = 1 + O(\bar{g}_0^{\,2}) \quad . \tag{92}$$

We want to use the renormalization group to evaluate Z beyond this power series form.

Just to get a feeling for what this means, lets ignore x_0 and σ_0 momentarily and suppose we had determined that $Z = 1 - z_0\bar{g}_0^{\,2}$. By studying the conditions for renormalizing γ_0 into γ and ν_0 into ν, Eqs. (37) and (40), we learn that

$$A(g) = \omega \frac{\partial}{\partial\omega}\bar{g}\,\bigg|_{\gamma_0,\nu_0} = -\frac{\varepsilon}{4}\bar{g} + a\bar{g}^3, \quad a>0 \quad . \tag{93}$$

Furthermore we note

$$H(\bar{g}) = \omega \frac{\partial}{\partial\omega}\log Z\,\bigg|_{\gamma_0,\nu_0} = h\,\bar{g}^2 \quad , \tag{94}$$

so

$$\frac{\partial}{\partial \bar{g}} \log Z(\bar{g}) = H(\bar{g})/A(\bar{g}) \quad . \tag{95}$$

Since $Z(\bar{g}=0) = 1$, we may integrate this to find

$$Z(\bar{g}) = \exp\int_0^{\bar{g}} dg' H(g')/A(g') \tag{96}$$

$$= (1 - \bar{g}^2/g_1^{\ 2})^{h/2a} \quad , \tag{97}$$

with

$$g_1^{\ 2} = \varepsilon/4a \quad . \tag{98}$$

In the same spirit the relation between $\bar{g}_0$ and $\bar{g}$ tells us

$$1 + \bar{g}_0^{\ 2}/g_1^{\ 2} = \frac{1}{1-\bar{g}^2/g_1^{\ 2}} \quad , \tag{99}$$

so

$$Z(\bar{g}_0) = (1 + \bar{g}_0^{\ 2}/g_1^{\ 2})^{-h/2a} \tag{100}$$

$$= 1 - \frac{2h}{\varepsilon} \bar{g}_0^{\ 2} + \cdots\cdots \tag{101}$$

This is the improvement on $Z(\bar{g}_0) = 1 - Z_0\bar{g}_0^{\ 2}$ which comes from using the renormalization group. The key ingredients were the power series in the renormalized coupling, g, and the inversion formula (99) to determine g as a function of g_0. This formula also shows an essential point of this exercise: as g^2 varies from zero to $g_1^{\ 2}$, $g_0^{\ 2}$ varies from zero to infinity. The form (100) for $Z(g_0)$ is valid in that whole range. This, as we will now see, allows us to treat large Reynolds number as well as small. As a last observation we remark that this procedure is systematically improvable by evaluating higher orders of perturbation theory for the renormalization group functions $A(g)$ and $H(g)$.

Now we want to determine the x_0 and σ_0 dependence of Z as well. For this we must enlarge upon our normalization conditions (36) and (37) as follows: choose

$$\frac{\partial}{\partial\omega} G_R^{(1,1)}(k^2,\omega)^{-1}\Big|_{\substack{\omega=i\omega_N\\ k^2=q_N^2}} = -i \tag{102}$$

and

$$\frac{\partial}{\partial k^2} G_R^{(1,1)}(k^2,\omega)^{-1}\Big|_{\substack{\omega=i\omega_N\\ k^2=q_N^2}} = \nu Z_1^{-1} \quad , \tag{103}$$

with Z_1 the scale factor relating $G_R^{(1,1)}$ and $G^{(1,1)}$

$$G^{(1,1)} = Z_1 G_R^{(1,1)} \quad , \tag{104}$$

and $G^{(1,1)}$ is the full (to all orders in γ_0) unrenormalized correlation function for χ_j and $\bar{\chi}_j$. So we must have

$$\frac{\partial}{\partial\omega} \; G^{(1,1)}(k^2,\omega,\gamma_0,\nu_0,k_0)^{-1}\Big|_{\substack{\omega=i\omega_N\\ k^2=q_N^2}} = -i\; Z_1(g,x,\sigma) \quad , \tag{105}$$

and

$$\frac{\partial}{\partial k^2} \; G^{(1,1)}(k^2,\omega,\gamma_0,\nu_0,k_0{}^2)^{-1}\Big|_{\substack{\omega=i\omega_N\\ k^2=q_N^2}} = \nu = \nu_0 Z_\nu(g,x,\sigma) \quad , \tag{106}$$

with

$$g = \frac{\gamma}{\nu^{3/2}}(q_N{}^2)^{(D-4)/4} \quad , \tag{107}$$

$$x = \frac{\nu q_N{}^2}{i\omega_N} \quad , \tag{108}$$

and

$$\sigma = q_N^{\,2}/k_0^{\,2} \quad . \tag{109}$$

γ is still given by the normalization condition (40), so Z_γ is as before, while Z_1 and Z_ν are defined by these revised normalization conditions. Before[3] we took $q_N^{\,2}=0$ and noted $Z_1=1$ then. The combination $\mathcal{Z} = Z_\gamma/Z_\nu^{3/2}$ relates g_0 and g as

$$g = \mathcal{Z} g_0 = \mathcal{Z} \frac{\gamma_0}{\nu_0^{\,3/2}} (q_N^{\,2})^{(D-4)/4} \quad . \tag{110}$$

Our goal is to find expressions for Z_ν and $\mathcal{Z}$ and from that for Z similar to the "improved" expression (100) given above. To begin we need the three renormalization group functions

$$A_\omega = \omega_N \frac{\partial}{\partial \omega_N} g \quad , \tag{111}$$

$$A_q = q_N^{\,2} \frac{\partial}{\partial q_N^{\,2}} g \quad , \quad \gamma_0, \nu_0 \text{ fixed}$$

and

$$A_0 = k_0^{\,2} \frac{\partial}{\partial k_0^{\,2}} g \quad ,$$

as well as

$$(B_\omega, B_q, B_0) = \left(\omega_N \frac{\partial}{\partial \omega_N}, q_N^{\,2} \frac{\partial}{\partial q_N^{\,2}}, k_0^{\,2} \frac{\partial}{\partial k_0^{\,2}}\right) \log Z_\nu \Bigg|_{\gamma_0, \nu_0 \text{ fixed}} \quad . \tag{112}$$

Next consider the various Z's as functions of g, x, σ which, using the chain rule, implies

$$\frac{A_\omega}{g} = A_\omega \frac{\partial}{\partial g} \log \mathcal{Z} + (B_\omega - 1) x \frac{\partial}{\partial x} \log \mathcal{Z} \quad , \tag{113}$$

$$\frac{A_q + \frac{\varepsilon}{4} g}{g} = A_q \frac{\partial}{\partial g} \log \mathcal{Z} + (1 + B_q) x \frac{\partial}{\partial x} \log \mathcal{Z} + \sigma \frac{\partial}{\partial \sigma} \log \mathcal{Z} \quad , \tag{114}$$

and

$$\frac{A_0}{g} = A_0 \frac{\partial}{\partial g} \log Z + B_0 \, x \frac{\partial}{\partial x} \log Z - \sigma \frac{\partial}{\partial \sigma} \log Z \quad . \tag{115}$$

From this we learn

$$g \frac{\partial}{\partial g} \log Z(g,x,\sigma) = \frac{\tilde{A}(g,x,\sigma) + \frac{\varepsilon}{4}\, g(1-B_\omega(g,x,\sigma))}{\tilde{A}(g,x,\sigma)} \quad , \tag{116}$$

with

$$\tilde{A}(g,x,\sigma) = (1-B_\omega)(A_0+A_q) + (1+B_0+B_q)A_\omega \quad . \tag{117}$$

The boundary condition

$$Z(g=0,x,\sigma) = 1 \tag{118}$$

allows us to write

$$Z(g,x,\sigma) = \exp\int_0^g \frac{dg'}{\tilde{A}(g',x,\sigma)} \left[\frac{\tilde{A}(g',x,\sigma)}{g'} + \frac{\varepsilon}{4}\,(1-B_\omega(g',x,\sigma))\right] . \tag{119}$$

Similarly we learn

$$Z_\nu(g,x,\sigma) = \exp\int_0^g dg' \, \frac{\tilde{B}(g',x,\sigma)}{\tilde{A}(g',x,\sigma)} \quad , \tag{120}$$

where

$$\tilde{B}(g,x,\sigma) = (1-B_\omega)(B_q+B_0) + (1+B_q+B_0)B_\omega \tag{121}$$

$$= B_0 + B_q + B_\omega \quad . \tag{122}$$

The same kind of equation for Z_1 is not needed unless we wish to evaluate the full $G^{(1,1)}$ by integrating (105) and (106). However, we do desire to have such a set of equations for Z so we need

$$C_\omega,\ C_q, C_o = \left(\omega_N \frac{\partial}{\partial \omega_N},\ q_N{}^2 \frac{\partial}{\partial q_N{}^2}, k_0{}^2 \frac{\partial}{\partial k_0{}^2}\right) \log\ Z(g,x,\sigma) \Bigg|_{\gamma_0,\nu_0} \tag{123}$$

and from this we discover

$$Z(g,x,\sigma) = \exp \int_0^g dg'\ \tilde{C}(g',x,\sigma) \Big/ \tilde{A}(g',x,\sigma) \quad , \tag{124}$$

where

$$\tilde{C}(g,x,\sigma) = (1-B_\omega)(C_0+C_q) + (1+B_q+B_0)C_\omega \quad . \tag{125}$$

Next we want to solve for g and x as functions of g_0 and x_0. To do that we must calculate the A, B, and C functions introduced above. We can only do this in perturbation theory in g and in this paper we will satisfy ourselves with the one loop graphs in Figs. 1, 2 and 3. The functions we need can be written as

$$\tilde{A}(g,x,\sigma) = -\frac{\varepsilon}{4} g + a(x,\sigma) g^3 \quad , \tag{126}$$

$$\tilde{B}(g,x,\sigma) = b(x,\sigma)\ g^2 \quad , \tag{127}$$

$$B_\omega(g,x,\sigma) = b_\omega(x,\sigma)\ g^2 \quad , \tag{128}$$

$$\tilde{C}(g,x,\sigma) = c(x,\sigma)\ g^2 \quad . \tag{129}$$

Using these in the equation for Z results in

$$Z(g,x,\sigma) = \left(1 - g^2/g_1(x,\sigma)^2\right)^{\frac{1}{2} - \frac{\varepsilon}{8} \frac{b_\omega(x,\sigma)}{a(x,\sigma)}} \quad , \tag{130}$$

with

$$g_1^{\,2}(x,\sigma) = \frac{\varepsilon}{4a(x,\sigma)} \quad , \tag{131}$$

$$Z_\nu(g,x,\sigma) = \left(1 - \frac{g^2}{g_1(x,\sigma)^2}\right)^{b(x,\sigma)/2a(x,\sigma)} \quad , \tag{132}$$

and

$$Z(g,x,\sigma) = \left(1 - \frac{g^2}{g_1(x,\sigma)^2}\right)^{c(x,\sigma)/2a(x,\sigma)} \quad . \tag{133}$$

From the Eqs. (113)-(115) and similar equations for derivatives of log Z and log Z_ν we can conclude that the ratios b_ω/a, b/a, and c/a entering in (130), (132), and (133) are independent[7] of x and σ . For example, take the derivative with respect to x of log Z

$$x \frac{\partial}{\partial x} \log Z = -\frac{\varepsilon}{8}\, x \frac{\partial}{\partial x} \frac{b_\omega(x,\sigma)}{a(x,\sigma)} \log\left(1 - \frac{g^2}{g_1(x,\sigma)^2}\right)$$
$$+ \left(\tfrac{1}{2} - \frac{\varepsilon b_\omega}{8a}\right) \frac{1}{1-g^2/g_1(x,\sigma)^2} \; \frac{2g^2}{g_1(x,\sigma)^3} \; x \frac{\partial}{\partial x} g_1(x,\sigma) \quad , \tag{134}$$

However, there will be no logarithms appearing in the expression for $x \frac{\partial}{\partial x} \log Z$, just ratios of polynomials, so b_ω/a is independent of x; much the same argument holds for its σ dependence, and the same holds for the ratios b/a and c/a. In any actual calculation we will not find explicit independence of these ratios except near $g = g_1$, but for purposes of carrying forward we will assume them constant and evaluated at $x=0$, $\sigma=0$. As a useful, but not necessary simplification, we will now imagine ε is small and use this in (130), so we may employ the relation

$$g_o = Z^{-1} g \tag{135}$$

to express g as a function of x, σ, and g_0

$$1 + g_0{}^2/g_1(x,\sigma)^2 = (1 - g^2/g_1(x,\sigma)^2)^{-1} \quad . \tag{136}$$

Then, of course,

$$Z_\nu = (1 + g_0{}^2/g_1(x,\sigma)^2)^{-b/2a} \quad , \tag{137}$$

and

$$Z = (1 + g_0{}^2/g_1(x,\sigma)^2)^{-c/2a} \quad . \tag{138}$$

The relation between x, g_0, and $x_0 = \nu_0 q_N{}^2/i\omega_N$ is given by

$$x = Z_\nu \, x_0 \tag{139}$$

$$= (1 + g_0{}^2/g_1(x,\sigma))^{-b/2a} \frac{\nu_0 q_N{}^2}{i\omega_N} \quad . \tag{140}$$

Since q_N and $i\omega_N$ were arbitrary points in $\vec{k},\omega$ space, we may finally write

$$Z(\bar{g}_0,\bar{x}_0,\bar{\sigma}_0) = \left(1 + \frac{\gamma_0{}^2}{\nu_0{}^3} \frac{(k^2)^{-\varepsilon/2}}{g_1(x,k^2/k_0{}^2)^2}\right)^{-c/2a} \quad , \tag{141}$$

with x determined implicitly by

$$x = \left[1 + \frac{\gamma_0{}^2}{\nu_0{}^3} \frac{(k^2)^{-\varepsilon/2}}{g_1(x,k^2/k_0^2)^2}\right]^{-b/2a} \frac{\nu_0 k^2}{\omega} \quad . \tag{142}$$

From these results we may now determine the behavior of $G^{(2,0)}$ for large and small k^2. For large k^2 we see that $Z \to 1$ and $G^{(2,0)}$ is just the lowest order value, the coefficient of Z in Eq. (88). This is what we expect because the effective Reynolds number as $k \to \infty$ goes to zero[3] and the non-linearity of the Navier-Stokes equation becomes unimportant.

For $k^2 \to 0$, we have

$$Z \underset{k\to 0}{\sim} \left[\frac{\nu_0{}^3 \; g_1^2(x,0)}{\gamma_0{}^2}\right]^{c/2a} (k^2)^{c\varepsilon/4a} \quad , \tag{143}$$

$$Z_\nu \underset{k\to 0}{\sim} \left[\frac{\nu_0{}^3 g_1{}^2(x,0)}{\gamma_0{}^2}\right]^{b/2a} (k^2)^{b\varepsilon/4a} \quad , \tag{144}$$

and from (142) we see that x is a function only of $\omega/(k^2)^{1+\varepsilon b/4a}$ for small k^2. This means $G^{(2,0)}$ behaves as

$$G^{(2,0)} \underset{k^2\to 0}{\sim} \frac{\Gamma_M(0)}{k^4} (k^2)^{\frac{\varepsilon}{4a}(c-2b)} F\left(\frac{\omega}{k^{2+\varepsilon b/2a}}\right) \quad , \tag{145}$$

while $E(k)$, the energy spectral function, which is

$$E(k) = \frac{2\pi^{D/2} k^{D-1}}{\Gamma(D/2)} (D-1) \int_{-\infty}^{+\infty} d\omega\, G^{(2,0)} \quad , \tag{146}$$

is

$$E(k) \underset{k\to 0}{\sim} k^{D-3-\rho} \quad , \tag{147}$$

where

$$\rho = -\frac{\varepsilon}{2a}(c-b) = -2g_1{}^2(c-b) \quad , \tag{148}$$

which is the same behavior found before. Because c/a and b/a are independent of x and σ , ρ will be $-\frac{1}{3}(4-D)$, when b and c are evaluated at D=4, as explained in Ref. 3.

We set out to find a method which allows us to determine the behavior of $G^{(2,0)}$ for all k^2, even for large Reynolds number. This Reynolds number can only be that formed from γ_0, ν_0, and k_0 - the physical unrenormalized parameters, since in $G^{(2,0)}(k^2,\omega,\gamma_0,\nu_0,k_0)$ only they appear; so it must be

$$R_0 = \frac{\gamma_0}{\nu_0{}^{3/2}} (k_0{}^2)^{(D-4)/4} \quad . \tag{149}$$

The equations for Z and x are

$$Z(\bar{g}_0,\bar{x}_0,\bar{\sigma}_0) = \left(1 + \frac{R_0^{\,2}}{g_1(x,k^2/k_0^{\,2})^2}\left(\frac{k_0^{\,2}}{k^2}\right)^{\varepsilon/2}\right)^{-c/2a} , \tag{150}$$

and

$$x = \left(1 + \frac{R_0^{\,2}}{g_1(x,k^2/k_0^{\,2})^2}\left(\frac{k_0^{\,2}}{k^2}\right)^{\varepsilon/2}\right)^{-b/2a} \frac{\nu_0 k^2}{\omega} . \tag{151}$$

These formulas hold for large and small R_0 - the physical Reynolds number. These now show us that we have achieved our aim. The question raised in the previous section is answered and the only physical length in the problem, namely, k_0^{-1}, sets the scale on which wave number variations may be examined.

Gathering together our formulae we finally find

$$G^{(2,0)}(\omega,k^2,R_0,\nu_0,k_0) = \frac{\Gamma_M(k^2/k_0^{\,2})\; x^2\; Z(R_0,x,k^2/k_0^{\,2})}{\nu_0 k^4(Z_\nu^{\,2}(R_0,x,k^2/k_0^{\,2})+x^2)} , \tag{152}$$

with x, Z_ν, and Z given before. This formula holds for all k and ω. The only approximation has been to use the one loop contribution to the renormalization group functions. This can be systematically improved by doing perturbation theory or some other summation procedure.[8] It is clear that a key ingredient in the whole analysis is the zero in $A(g)$ at $g_1^{\,2} \approx \varepsilon$. This is the important observation that enables us to use our formulae for all k and allows us to carry out the resummation of perturbation theory expressed in (150). We have demonstrated this zero only to one loop order in perturbation theory in g. Experience in condensed matter studies of phase transitions and high energy physics indicates that this zero will persist in higher order evaluations of $A(g)$.

One immediate implication of this whole exercise is that we see nowhere the natural emergence of a $k^{-5/3}$ law for the energy spectral function. It would seem appropriate to conclude that when we maintain turbulence by a random mixing—stationary turbulence—in the homogeneous, isotropic situations we have examined, the reasoning—almost dimensional analysis[4]—leading to the $k^{-5/3}$ law may not be correct. It is possible now to use (152) for a specific choice of $\Gamma_M(k^2/k_0{}^2)$ to study the behavior of E(k) in a situation where R_0 is very large and the inertial range: $k \ll k \ll \sqrt{R_0}\, k_0$ is also sizeable. We know from the general analysis that for very small k, $E(k) \sim k^{-\rho}$ at D=3 and for very large k $E(k) \sim \Gamma_M(k^2/k_0{}^2)$ at D=3. Somewhere in between it must stop rising as $k^{-\rho}$ and then turn over to fall as Γ_M. For some range it may well behave as $k^{-5/3}$, but the present analysis gives little reason to focus on that power law.

VI. SUMMARY AND DISCUSSION

In this paper we have treated a turbulent fluid mixed by a random stirring force which sets up a stationary homogeneous, and isotropic velocity field. Although such a physical circumstance is not normally encountered it is not difficult to imagine creating such a situation in a laboratory environment. The requirement of stationarity means that we are asking for the response of a fluid obeying the Navier-Stokes equation to an external random forcing set up very long ago and continuing into the future. The issue then is the importance of the non-linearity due to the $\vec{v}\cdot\nabla v_j$ term in the dynamics. We find that for a wide class of forcing functions the non-linearity is always weak in spatial dimensions less than four. These forcing functions are those whose correlation does not vanish at vanishing wave number. Using the Novikov[5] connection between for correlation function of the mixing and the net energy dissipation due to viscosity we argued that this behavior is likely to hold in many physical situations.

After the discussion of turbulence mixed essentially in small wave number ($k \leq k_0$ = wave number characterizing the external stirring) we turned to the construction of the two fold velocity correlation function

$$\Phi^{(2,0)}_{j\ell}(\vec{k},\omega) = \int d^D x\ e^{i(\vec{k}\cdot\vec{x}-\omega t)} \langle v_j(\vec{x},t)\, v_\ell(0,0)\rangle \quad , \qquad (153)$$

using the renormalization group. Our technique, adopted from similar work in high energy physics,[6,7] consists in essence of summing pieces of all orders of perturbation theory in the dimensionless coupling of the theory, the Reynolds number. In one view it adds up all the most singular terms near four space dimensions for each order of perturbation theory. The procedure is systematically improvable by making better approximations to the renormalization group functions. We used the one loop approximation in this work and can imagine doing two or three loops with some effort. After that one can turn to recent techniques[8] for learning the behavior of such functions in very high order of perturbation theory to underpin a Borel or Padé-Borel approximation method.

In the mixing forces we treated, the energy spectral function $E(k)$ behaves at small k as $k^{D-3-\rho}$ where ρ is determined as a power series in $\rho=4-D$ and in the lowest order is $\rho=-1(4-D)/3$ as in our previous work.[3] For large wave number, $E(k)$ follows the input spectrum accurately since the effective Reynolds number behaves as $k^{-(4-D)/4}$ for $D<4$ and non-linearities are not important. In the middle range one may study in detail any particular model for the stirring by the methods indicated just above. There does not seem to be any clear way in which a Kolgomorov $k^{-5/3}$ spectrum appears.

A much more realistic example of mixing can be treated with our methods. A fluid passing through a screen of spacing M may be represented, from the viewpoint of an observer moving with the mean flow, as a stirring occurring primarily at wave number $\lesssim M^{-1}$ and during a very brief time interval. A random forcing function would then represent the perturbations suffered by the flow and be a stand in for a random distribution of velocities taken at the screen. If the duration of mixing were T, a suitable correlation function for the mixing may be

$$\langle F_j(\vec{x},t)\ F_\ell(\vec{y},\tau)\rangle = \frac{\gamma^2}{4}\,\Delta_{j\ell}(\nabla)\,\Gamma_M((\vec{x}-\vec{y})^2/M^2)\,\delta(\tau-t)\times\theta(t)\,\theta(T-t)\ . \quad (154)$$

The study of the turbulent motion after such a pulsed mixing will be presented in the next paper in this series.

NOTE ADDED IN MANUSCRIPT:

After writing this paper I received a communication in which it was kindly brought to my attention by U. Frisch that there is a body of work on the use of the renormalization group in the theory of turbulence that had escaped my attention. This work is summarized in the thesis of Jean-Daniel Fournier entitled "Introduction to the Renormalization Group for the Study of Certain Problems of Large Scale Turbulence" (in French). Particular note appears appropriate of the work of D. Forster, D.R. Nelson, and M.J. Stephen, Phys. Rev. A16, 732 (1977) and J.-D. Fournier and U. Frisch, Phys. Rev. A17, 747 (1978).

These authors use the renormalization group in the fashion of J. Kogut and K. Wilson, Phys. Rep. 12C, 75 (1974), which is very well tuned to the study of the small wave number behavior of turbulence. The intermediate and large wave number behavior and the explicit construction of the correlation functions, as is done in the present paper, are quite difficult in that approach. Clearly the spirit is in this earlier work and must be properly acknowledged. I am very grateful to Professor Frisch for making me aware of this work.

REFERENCES

[1]E. Hopf, J. Rat. Mech. Anal. 1, 87 (1962). A good exposition of the Hopf equation and its connection with field theory may be found in Chapter 10 of A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence (The MIT Press, Cambridge, Mass., 1975), Vol. II. See also V.I. Tatarskii, Soviet Physics-JETP 15, 961 (1962) and H.W. Wyld, Jr., Ann. Phys. (N.Y.) 14, 143 (1961).

[2]S.F. Edwards, J. Fluid Mech. 18, 239 (1964) and R.H. Kraichman, J. Fluid Mech. 83, 349 (1977). Also D.C. Leslie, Rep. Prog. Phys. 36, 1365 (1973).

[3]H.D.I. Abarbanel, Fermilab Preprint THY-78/28, March, 1978

[4]G.K. Batchelor, The Theory of Homogeneous Turbulence, 2nd Edition, (Cambridge University Press, London, 1971).

[5]E.A. Novikov, Soviet Physics-JETP 20, 1290 (1965).

[6]W.R. Frazer, H. Hoffman, J.R. Fulco, and R.L. Sugar, Phys. Rev. D14, 2387 (1976).

[7]H.D.I. Abarbanel, J. Bartels, J.B. Bronzan, and D. Sidhu, Phys., Rev. D12, 2798 (1975).

[8]It is now known that the perturbation series for such functions is divergent but is Borel-summable, so excellent approximations to them may, in fact be determined. See L.N. Lipatov, Soviet Physics-JETP 45, 216 (1977), and for a thorough expositon see E. Brézin, J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. D15, 1544 (1977).

FIGURE CAPTIONS

Fig. 1 The graphs, up to one loop, contributing to the velocity—anti-velocity correlation function. These give the renormalization constant Z_ν which rescales the viscosity.

Fig. 2 The graphs, up to one loop, contributing to the fusion vertex which is the non-linearity of turbulent motion. These give the renormalization constant Z_γ which rescales the non-linearity.

Fig. 3 The graphs, up to one loop, contributing to the velocity—velocity correlation functions.

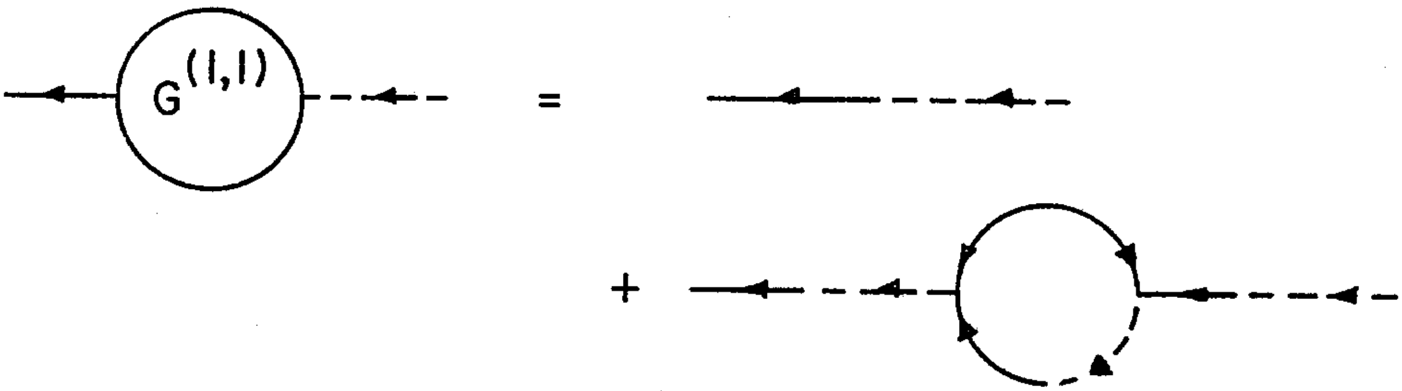

Fig. 1

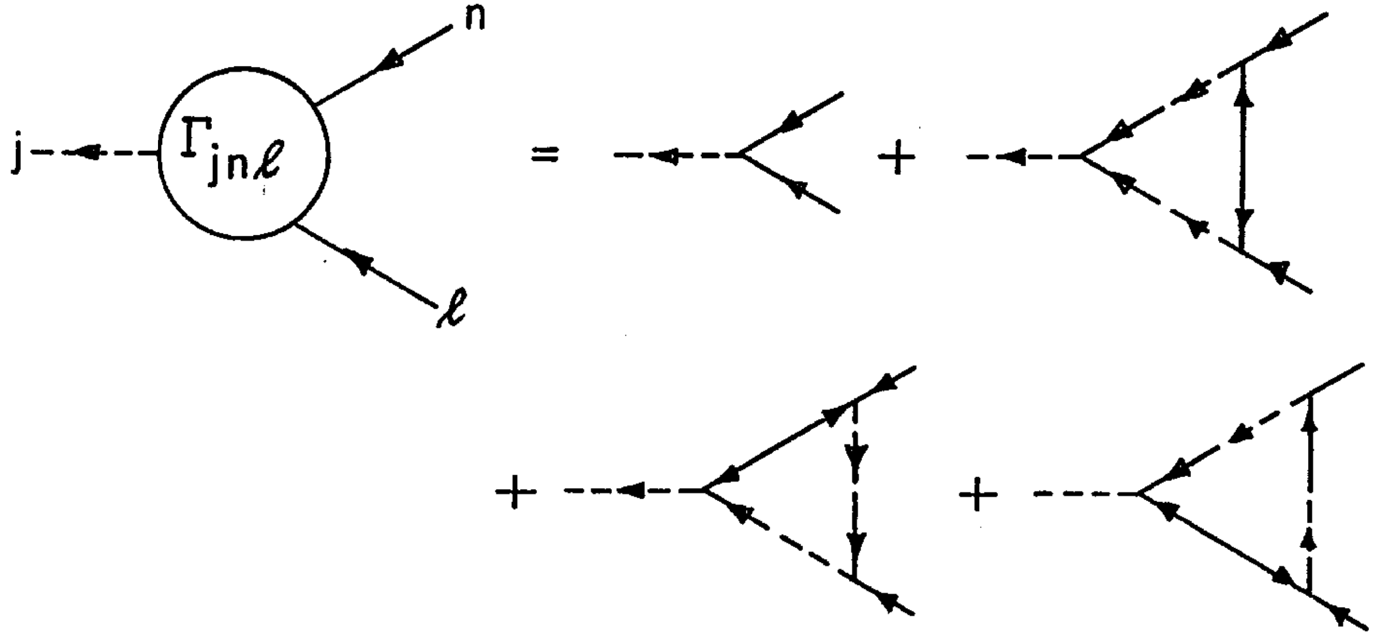

Fig. 2

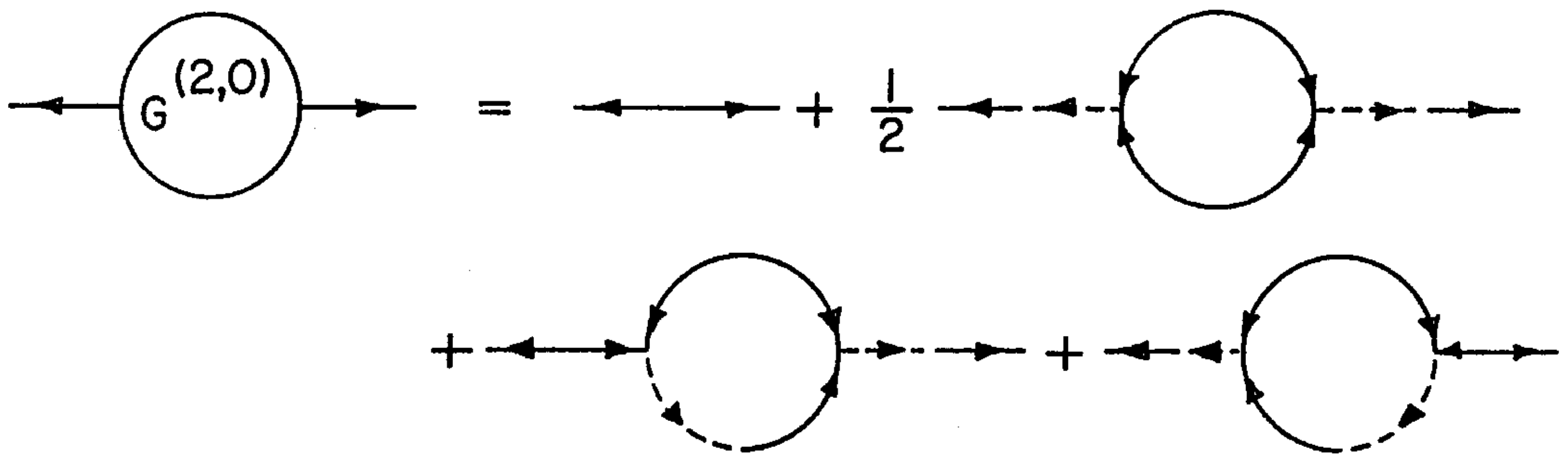

Fig. 3