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Dynamics of Non-linear Stochastic Systems

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I. INTRODUCTION

In this set of lectures I wish to bring together a variety of problems and techniques from various subjects concerned with non-linear stochastic systems. These include quantum field theory, wave propagation through a random medium, turbulent fluid flow, radar scattering from the sea surface, and others.

A goal of the lectures is to discuss the issues on a more-or-less unified footing and bring to bear on them a technique, called the renormalization group method, which allows a quantitative assessment of the importance of the non-linearities of the problem in question. Under some circumstances the same method allows one to construct the correlation functions for the fluctuating variables of interest. All of this will be explained in some detail as we proceed and will presuppose no knowledge of the renormalization group as used either in quantum field theory or in statistical physics.

Our plan is to begin with some familiar and often well documented examples and build on that. In the first lecture we start with consideration of a scalar wave propagating through a fluctuating medium and demonstrate how this is directly related to a particular "quantum" field theory. A similar task is carried out for diffusion of a scalar field in the presence of a random fluctuation.

For these examples we then discuss in detail the renormalization group analysis of their correlation functions.

After these examples are considered in detail, we proceed to use the same tools to study turbulent fluid motion. We begin with a discussion of the time dependent problem of homogeneous, isotropic turbulence, then consider in some detail stationary, homogeneous, isotropic turbulence. A return to the decay of turbulent motion is made at the end.

There are several references which have extensive treatments of various bits and pieces of the material here. I encourage the reader not to feel compelled to read them all in great detail, but to taste each for the flavor:

V.I. Tatarskii, The Effects of the Turbulent Atmosphere on Wave Propagation, available from NITS, 1971

V. Frisch, "Wave Propagation in Random Media," in Probabilistic Methods in Applied Mathematics, Vol. 1, edited by A.T. Bharuchi-Reid, p. 75-198 (Academic Press, N.Y., 1968).

Monin, A.S. and A.M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. II (MIT Press, Cambridge, Mass., 1975).

Bjorken, J.D. and S.D. Drell, Relativistic Quantum Fields (Mc Graw-Hill, N.Y., 1965).

Kogut, J. and K. Wilson, "The Renormalization Group and the ε-expansion," Phys. Rep. 12C, 75 (1974).

Abers, E.S. and B.W. Lee, Gauge Theories, Phys. Rep. 9C, 144 (1973);

Brezin, E., J.C. Le Guillou, and J. Zinn-Justin in Phase Transitions and Critical Phenomena, VI, C. Domb and M.S. Green, editors (Academic Press, N.Y., 1976).

LECTURE 1 Motion in a Random Medium as a Non-Linear Field Theory

We'll begin with a discussion of a scalar wave $\psi(\vec{x},t)$ propagating through a medium with a random index of refraction or random sound speed. Let's take here a medium with wave speed

$$c(\vec{x},t) = c_0 + \delta c(\vec{x},t) \tag{1}$$

with $\delta c \ll c_0$ everywhere. The wave received at $\vec{x}$,t from a source at $\vec{x}_0$ with a signal s(t) is given by

$$\left(\nabla^2 - \frac{1}{c(\vec{x},t)^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{x},t) = -\delta^3(\vec{x}-\vec{x}_0)\,s(t); \tag{2}$$

because δc is small we may write this as

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} + \frac{2\,\delta c(\vec{x},t)}{c_0}\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{x},t) = -\delta^3(\vec{x}-\vec{x}_0)\,s(t), \tag{3}$$

The piece $\delta c(\vec{x},t)$ of the velocity is assumed to be a fluctuating quantity, while c_0 is a constant, or, in the ocean sound channel, a determined function of space and time.

Let's now take the source to be mono-chromatic, $s(t) = e^{-i\omega t}$, and write $\psi(\vec{x},t) = \phi(\vec{x})e^{-i\omega t}$. When the variations in $\delta c(\vec{x},t)$ are slow compared to the $e^{-i\omega t}$ of the source, we make the slow medium approximation by neglecting the time-dependence of δc.

Furthermore we'll assume c_0 is constant so the medium without δc is homogeneous. The wave equation now becomes

$$\left(\nabla^2 + k_0^2 - F(\vec{x})\right) \phi(\vec{x}) = -\delta^3(\vec{x} - \vec{x}_0) \qquad (4)$$

with the fluctuation operator

$$F(\vec{x}) = \frac{2\,\delta c(\vec{x})}{c_0} k_0^2 \,, \quad k_0 = \frac{\omega}{c} \,. \qquad (5)$$

This is the prototypical equation for much of our subsequent discussion. It has been treated at great length by Frisch (1968), Tatarskii (1971), Mysak (1978), Barabanenkov, Kravtsov, Tatarskii, and Rytov (1971), and many others. Since we have a pedagogical goal here, much of the material found in those references will be repeated.

I will occasionally employ an operator notation for equations like (4) and write

$$\left(G_0^{-1} - F\right) \phi = -1 \qquad (6)$$

The meaning of G_0^{-1} is nothing by $\nabla^2 + k_0^2$ when acting on functions of $\vec{x}$. G_0 is the inverse of that and solves the $\vec{x}$ space equation

$$\left(\nabla^2 + k_0^2\right) G_0(\vec{x}) = \delta^3(\vec{x}). \qquad (7)$$

Henceforth, I will assume the medium to be unbounded and the wave $\phi(\vec{x})$ to vanish as $|\vec{x}| \to \infty$. The condition of outgoing radiation from $\vec{x}_0$ leads, as usual, to

$$G_0(\vec{x}) = \int \frac{d^D q}{(2\pi)^D} \frac{e^{i\vec{q}\cdot\vec{x}}}{k_0^2 - \vec{q}^{\,2} + i\varepsilon} \tag{8}$$

$$= -\frac{1}{4\pi} \frac{e^{ik_0|\vec{x}|}}{|\vec{x}|} \quad \text{at} \quad D = 3. \tag{9}$$

Now the solution of (6) is

$$\phi = -\left(G_0^{-1} - F\right)^{-1} \tag{10}$$

$$= -\frac{1}{1 - G_0 F} G_0 = -G_0 \frac{1}{1 - F G_0}. \tag{11}$$

This has the expansion

$$\phi = -\left(G_0 + G_0 F G_0 + G_0 F G_0 F G_0 + \cdots\right) \tag{12}$$

which provides the formal solution to the problem at hand.

The physical question, however, is not ϕ, but what field $\phi(\vec{x})$ is received at $\vec{x}$ when properties of the medium are averaged over. We call this $\langle \phi(\vec{x}) \rangle$. Since $F(\vec{x})$ is a random function, we must specify that <u>probability</u> <u>distribution</u> <u>functional</u> for it. For this, suppose first that $F(\vec{x})$ is specified not as a function over all labels $\vec{x} = (x_1, x_2, \cdots x_D)$ of some D dimensional co-ordinate space with $-\infty < x_i < \infty$, but at a discrete set of points $j=1,2,\cdots N$ in that space. Now these F_j are a set of random variables. At every j we have a random variable which ranges over $-\infty<F_j<+\infty$ with some joint probability distribution $P[F_1, F_2, \cdots F_N]$. As a concrete example, suppose $P[F_j]$ is gaussian

$$P[F_j] = \tilde{N}\, exp\left[-\frac{1}{2} \sum_{j,k=1}^{N} F_j M_{jk} F_k\right] \qquad (13)$$

with $\tilde{N}$ a normalization factor arranged so

$$\int dF_1 \cdots dF_N\, P[F_j] = 1 \qquad (14)$$

Since

$$\int \prod_{j=1}^{N} dF_j\, e^{-\frac{1}{2}\sum\limits_{k,\ell} F_k m_{k\ell} F_\ell} = \frac{(2\pi)^{N/2}}{(det\, m)^{1/2}}\,, \qquad (15)$$

$$\tilde{N} = (det\, m)^{1/2}\, (2\pi)^{-N/2}. \qquad (16)$$

Generally we won't specify $\tilde{N}$ since the quantities of real interest are averages of functions of the variables F_j. These are given as

$$\left\langle \mathcal{F}(F_1, \ldots F_N)\right\rangle = \frac{\int dF_1 \cdots dF_N\, \mathcal{F}(F_1, \cdots F_N)\, e^{-\frac{1}{2}\sum\limits_{j,\ell=1}^{N} F_j M_{j\ell} F_\ell}}{\int dF_1 \cdots dF_N\, e^{-\frac{1}{2}\sum\limits_{j,\ell=1}^{N} F_j M_{j\ell} F_\ell}}\,, \qquad (17)$$

for our gaussian (13), and $\tilde{N}$ cancels out.

Furthermore, we will often be interested in averages of polynomials in the F_j, and for this it is extremely convenient to construct the generating function

$$Z[J_1, \cdots J_N] = \int dF_1 \cdots dF_N \; P[F_1, \cdots F_N] \; e^{\sum_{\ell=1}^{N} F_\ell J_\ell} . \qquad (18)$$

If we know this, then

$$\langle F_k \rangle = \int dF_1 \cdots dF_N \; F_k \; P[F_j] \qquad (19)$$

$$= \left. \frac{\partial Z[J_\ell]}{\partial J_k} \right|_{J_\ell = 0} . \qquad (20)$$

and other averages are given by the appropriate derivatives. (If $J_\ell = i \varphi_\ell$, the function $Z[\varphi_n]$ is known as the characteristic function for $P[F_i]$). In our gaussian example, we can evaluate $Z[J]$ directly and find

$$Z[J_i] = exp \frac{1}{2} \sum_{\ell, k=1}^{N} J_\ell \Gamma_{\ell k} J_k \qquad (21)$$

where the matrix Γ is the inverse of M: $\sum_{j=1}^{N} M_{ij} \Gamma_{j\ell} = \delta_{i\ell}$.
From this we find for the moments in a gaussian distribution

$$\langle F_j \rangle = 0 , \qquad (22)$$

$$\langle F_j F_\ell \rangle = \Gamma_{j\ell} , \; etc \qquad (23)$$

which is all quite familiar.

Returning to our medium we wish to let the label j on F_j become a continuous index $\vec{x}$. This now emphasizes the role played by space as a purely labeling, non-dynamic quantity in field theory. Then our function $P[F_j]$ becomes a functional $P[F(\vec{x})]$. The limiting procedure requires a certain level of careful mathematics to allow it to be presentable, but no mistakes have ever been made to my knowledge by simply considering $\vec{x}$ to be the limit of a bunch of points $j=1,2,\cdots N$ as $N \to \infty$ and the spacing goes to zero.

For a gaussian we would write

$$P[F(\vec{x})] = \eta \exp -\frac{1}{2} \int d^D x\, d^D y\, F(\vec{x})\, M(\vec{x},\vec{y})\, F(\vec{y}) , \qquad (24)$$

where properly speaking η is not now well defined. I urge you to ignore this, carry η around if you like, and watch it cancel out of physically sensible quantities. The generating functional for this gaussian distribution is

$$Z[J(\vec{x})] = \exp \frac{1}{2} \int d^D x\, d^D y\, J(\vec{x})\, \Gamma(\vec{x},\vec{y})\, J(\vec{y}) \qquad (25)$$

with

$$\int d^D z\, M(\vec{x},\vec{z})\, \Gamma(\vec{z},\vec{y}) = \delta^D(\vec{x}-\vec{y}) . \qquad (26)$$

An average of $F(\vec{x})$ or any powers, $F(\vec{x}_1)\cdots F(\vec{x}_M)$, comes from $Z[J(\vec{x})]$

$$Z[J(\vec{x})] = \int \prod_{\vec{x}} dF(\vec{x}) \; e^{\int d^D u \, J(\vec{u}) F(\vec{u})} \; P[F(\vec{u})], \qquad (27)$$

by functional differentiation; for example,

$$\langle F(\vec{y}) \rangle = \left. \frac{\partial Z[J(\vec{x})]}{\partial J(\vec{y})} \right|_{J=0} = \int \prod_{\vec{x}} dF(\vec{x}) \; F(\vec{y}) \; P[F(\vec{u})]. \qquad (28)$$

These are just the continuum versions of (18) - (20). The integrals in (27) and (28) are integrals over $-\infty < F < \infty$ at every labeling point $\vec{x}$ of the D dimensional space on which the field $F(\vec{x})$ lives.

Returning once again to the gaussian (24) we have

$$\langle F(\vec{x}) \rangle = 0 \qquad (29)$$

and

$$\langle F(\vec{x}) \, F(\vec{y}) \rangle = \Gamma(\vec{x}, \vec{y}). \qquad (30)$$

One last point: if the medium over which $F(\vec{x})$ is defined is homogeneous, then $\langle F(\vec{x}_1) \cdots F(\vec{x}_n) \rangle$ can only depend on the n-1 differences $\vec{x}_1 - \vec{x}_2 \; \cdots \; \vec{x}_n - \vec{x}_{n-1}$. This comes about because we could translate each point $\vec{x}_i$ by $\vec{a}$ and the invariance of the medium under this means

$$\langle F(\vec{x}_1) \cdots F(\vec{x}_n) \rangle = \langle F(\vec{x}_1 + \vec{a}) \cdots F(\vec{x}_n + \vec{a}) \rangle. \qquad (31)$$

Choose $\vec{a} = -\vec{x}_n$, say, and it is clear this average depends only on $\vec{x}_1 - \vec{x}_n, \; \cdots \; \vec{x}_{n-1} - \vec{x}_n$. Physically this means that there are no boundaries to the region in which we work, for translating by $\vec{a}$ would not yield the same averages for $F(\vec{x}_1)\cdots F(\vec{x}_n)$ if we hit or went out of the boundaries. In a real situation where there are inevitably walls, this translation invariance will usually be a good physical property when discussing functions "far away" from the walls.

After this long digression we can come back to the solution (11) or (12) for the scalar field ϕ felt at $\vec{x}$ and ask for its average over realizations of F:

$$\langle\phi(\vec{x})\rangle = \int \prod_{\vec{x}} dF(\vec{x})\; P[F(\vec{x})]\,\phi(\vec{x}) \tag{32}$$

$$= -\int \prod_{\vec{x}} dF(\vec{x})\, P[F(\vec{x})]\,\left(G_0 + G_0 F G_0 + G_0 F G_0 F G_0 + \cdots\right). \tag{33}$$

For our gaussian distribution (24) we can write out the explicit value of the first few terms

$$\langle\phi(\vec{x})\rangle = -G_0(\vec{x}-\vec{x}_0) - \int d^D y\, d^D z\; G_0(\vec{x}-\vec{y})\,\Gamma(\vec{y},\vec{z})\,G_0(\vec{y}-\vec{z})\,G_0(\vec{z}-\vec{x}_0) + \cdots \tag{34}$$

using (29) and (30).

It is important to give in words the meaning of the series (12) or (34). The series (12) represents multiple scattering of the wave from the source at $\vec{x}_0$ on its way to the receiver at $\vec{x}$. (See Figure 1 where the quadratic term in F is represented). Waves emitted from $\vec{x}_0$ may arrive at

$\vec{x}$ without interacting with a fluctuation F: this is $\phi = -G_0$; it may interact once: this is $-G_0FG_0$, since G_0 brings the wave from $\vec{x}_0$ to the local fluctuation F and then G_0 carries it to $\vec{x}$; etc.

If the medium were not fluctuating, we could say no more. However, F fluctuates, and, what is essential, these fluctuations are correlated, since generally

$$\langle F(\vec{x})\, F(\vec{y}) \rangle \neq 0 . \tag{35}$$

A fluctuation, therefore, at $\vec{x}$ "communicates" with a fluctuation at $\vec{y}$, and a wave which scatters off $F(\vec{x})$ <u>interacts with itself</u>, when it arrives at $F(\vec{y})$. The mechanism of this self interaction is precisely the correlation of the fluctuations in the medium. The series (34) expresses all this explicitly for our gaussian distribution of fluctuations. Since the mean fluctuation is zero, one interaction of the wave with F averages to zero.

Two interactions with F makes a very different story. Representing Equation (34) in pictures in Figure 2 with G_0 represented by a solid line and Γ by a wiggly line, we can think of the 2nd term as a wave which arrives at $\vec{z}$ and interacts with $F(\vec{z})$. When it now goes over to $\vec{y}$ and interacts with $F(\vec{y})$, that second interaction "knows" about the first because of the correlation in the fluctuations. The self interaction represents the information in the fluctuating medium that a wave arriving at $\vec{y}$ got there via $\vec{z}$ and those two points are correlated. It is this self interaction which is responsible for the non-linearity of the process of wave propagation in a random medium.

As an aside at this point we should note that precisely these words apply to a quantum field theory such as electrodynamics. There, for example, electrons (waves) propagate through the vacuum which is populated by fluctuations whose origin lies in the uncertainty principle. The fluctuations are correlated by the propagation of another wave (photons or light in electrodynamics) in just the way indicated by Γ in Equation (34) or the wiggly lines in Figure 2. In a quantum field theory, however, the fluctuations are themselves dynamic and do not generally have a prescribed distribution functional. So there we have the interaction of two dynamical fields. In propagation through a random medium, the medium is taken to fluctuate in a prescribed fashion - given by $P\left[F(\vec{x})\right]$ - and the passage of the wave is assumed not to disturb the medium by its interaction with it. The origin of the stochastic behavior in quantum field theory is the non-zero fluctuations of all co-ordinates because $\hbar \neq 0$. In our wave propagation problem, the stochastic behavior comes from the specified fluctuating medium. Because the distribution functional for one of the fields - the medium $F(\vec{x})$ - is given in the wave propagation situation, the problem is easier that in the case of interacting quantum fields, but what is quite essential is the fact that otherwise the problems are identical. We will exhibit this explicitly below. It is no surprise, therefore, that techniques of analysis and approximation carry directly from one to the other.

I want next to develop a formalism which exhibits the non-linearity inherent in the self-interacting scalar wave propagation. To begin we consider F to be non-fluctuating. The amplitude to receive a signal at $\vec{x}_1, \cdots \vec{x}_N$ from a source at $\vec{x}_0$ is just the product

$$\phi(\vec{x}_1) \cdots \phi(\vec{x}_N) = G(\vec{x}_1, \vec{x}_0) \cdots G(\vec{x}_N, \vec{x}_0) \tag{36}$$

where G is the operator $-(G_0{}^{-1}-F)^{-1}$. The function $G(\vec{x}_a, \vec{x}_b)$ is like the element of a matrix, G, in a space labeled by the continuous indices $\vec{x}$. Each factor G propagates a wave- independently of the other waves - from $\vec{x}_0$ to the appropriate receiver.

We want to write $G(\vec{x}_a, \vec{x}_b)$ as a gaussian integral over an uxiliary field $\chi(\vec{x})$. For this purpose we note from (24) and (30)

$$G(\vec{x}_a, \vec{x}_b) = \eta \int \prod_{\vec{x}} d\chi(\vec{x}) \;\; \chi(\vec{x}_a)\chi(\vec{x}_b)\, exp - \frac{1}{2}\int d^Dx\, d^Dy\; \chi(\vec{x})\, G^{-1}(\vec{x}, \vec{y})\, \chi(\vec{y}) \tag{37}$$

with

$$\eta^{-1} = \int \prod_{\vec{x}} d\chi(\vec{x})\; exp - \frac{1}{2}\int d^Dx\, d^Dy\; \chi(\vec{x})\, G^{-1}(\vec{x}, \vec{y})\, \chi(\vec{y}) \tag{38}$$

so

$$G(\vec{x}_a, \vec{x}_b) = \langle \chi(\vec{x}_a)\chi(\vec{x}_b) \rangle_\chi \tag{39}$$

Formally η = (infinite constant) $(\det G)^{1/2}$. We want to average this over the distribution of F and for convenience we wish to eliminate the factor η and have F appear only in the exponential of (37). We utilize a trick here which has been employed in the study of electrons moving in a metal with random impurities. Generalize $\chi(\vec{x})$ to a field with M components: $\chi_\alpha(\vec{x})$ $\alpha = 1, 2, \cdots M$, and consider the integral

$$\int \prod_{\vec{u}\,\alpha=1}^{M} d\chi_\alpha(\vec{u}) \;\; \chi_\lambda(\vec{w})\chi_\sigma(\vec{y}) \exp -\frac{1}{2} \int d^Dx\, d^Dz \sum_{\alpha=1}^{M} \chi_\alpha(\vec{x})\, G^{-1}(\vec{x},\vec{z})\, \chi_\alpha(\vec{z}) \tag{40}$$

$$= \delta_{\lambda\sigma} \;\; \eta^{-M}\, G(\vec{w},\vec{y}). \tag{41}$$

This leads to

$$G(\vec{w},\vec{y}) = \lim_{M\to 0} \int \prod_{\substack{\alpha=1 \\ \vec{u}}}^{M} d\chi_\alpha(\vec{u})\;\; \chi_1(\vec{w})\chi_1(\vec{y}) \exp -\frac{1}{2} \int d^Dx\, d^Dz \sum_{\alpha=1}^{M} \chi_\alpha(\vec{x})\, G^{-1}(\vec{x},\vec{z})\, \chi_\alpha(\vec{z}), \tag{42}$$

where the limit $M \to 0$ is to be taken after the integration is performed. Let's look at the argument in the exponential more closely. It is an action

$$\text{Action} = \int d^Dz \;\frac{1}{2} \sum_{\beta=1}^{M} \left[\left(\nabla\chi_\alpha(\vec{z})\right)^2 - k_0^2\left(\chi_\alpha(\vec{z})\right)^2 - F(\vec{z})\left(\chi_\alpha(\vec{z})\right)^2 \right] \tag{43}$$

$$= \int d^Dx \;\; \mathcal{L}(\chi_\alpha)$$

If $\chi_\alpha(\vec{x})$ were an ordinary field, with the action, it would satisfy the Euler-Lagrange equation

$$\nabla_j \left(\frac{\partial \mathcal{L}(\chi_\alpha)}{\partial(\nabla_j \chi_\alpha(\vec{x}))} \right) = \frac{\partial \mathcal{L}}{\partial \chi_\alpha(\vec{x})} \tag{44}$$

or

$$\left[\nabla^2 + k_0^2 - F(\vec{x})\right] \chi_\alpha(\vec{x}) = 0 \tag{45}$$

We can regard $\chi_\alpha(\vec{x})$ as a kind of pseudo-stochastic field with a gaussian distrubition.

Now we are ready to average $G(\vec{x}_1, \vec{x}_0)$ over the fluctuations of F

$$\langle \phi(\vec{x}_1) \rangle = \int \prod_{\vec{x}} dF(\vec{x})\, P[F(\vec{x})] \int \prod_{\substack{\alpha=1 \\ \vec{u}}}^{M} d\chi_\alpha(\vec{u})\, \chi_1(\vec{x}_1) \chi_1(\vec{x}_0)\, e^{-[\text{Action}]}, \tag{46}$$

where the limit $M \to 0$ is understood at the end of the calculation.

If $P[F(x)]$ is a gaussian, we can do the integral over F directly to find

$$\langle \phi(\vec{x}_1) \rangle = \int \prod_{\substack{\alpha=1 \\ \vec{u}}}^{M} d\chi_\alpha(\vec{u}) \chi_1(\vec{x}_1) \chi_1(\vec{x}_0)\, \exp - \int d^D z\, \mathcal{L}_{eff}(\chi_\alpha(\vec{z})) , \tag{47}$$

where

$$\mathcal{L}_{eff}(\chi_\alpha(\vec{z})) = \frac{1}{2} \sum_{\alpha=1}^{M} \left[(\nabla \chi_\alpha(\vec{z}))^2 - k_0^2 (\chi_\alpha(\vec{z}))^2 \right] - \frac{1}{8} \sum_{\alpha=1}^{M} (\chi_\alpha(\vec{z}))^2 \int d^D y\, \Gamma(\vec{z}, \vec{y}) \sum_{\beta=1}^{M} (\chi_\beta(\vec{y}))^2 . \tag{48}$$

This is really the essential result of this section of the lectures. It gives the average signal received at $\vec{x}_1$ from a point source at $\vec{x}_0$ in terms of an integral representation. It makes quite explicit the nature of the non-linearity induced by the medium.

If $F(\vec{x})$ is not gaussian, then we recognize the generating functional

$$Z_F[J(\vec{x})] = \int \prod_{\vec{u}} dF(\vec{u})\, P[F(\vec{u})]\, \exp \int d^D y\, J(\vec{y}) F(\vec{y}) \qquad (49)$$

entering (4) with $J(\vec{y}) = \frac{1}{2}\sum_{\alpha=1}^{M} (\chi_\alpha(\vec{y}))^2$. The effective lagrangian density for this case is

$$\mathcal{L}_{eff} = \frac{1}{2}\sum_{\alpha=1}^{M} [(\nabla\chi_\alpha)^2 - k_0^2(\chi_\alpha)^2] - \log Z_F\left(\frac{1}{2}\sum_{\alpha=1}^{M}(\chi_\alpha)^2\right). \qquad (50)$$

To find the signal received at $\vec{x}_1, \cdots \vec{x}_N$ from the source at $\vec{x}_0$ we desire

$$\langle G(\vec{x}_1, \vec{x}_0) \cdots G(\vec{x}_N, \vec{x}_0) \rangle. \qquad (51)$$

The product of Green functions is

$$G(\vec{x}_1,\vec{x}_0)\cdots G(\vec{x}_N,\vec{x}_0) = \lim_{M\to 0} \int \prod_{\substack{\alpha=1 \\ \vec{u}}}^{M} d\chi_\alpha(\vec{x}) \{\chi_1(\vec{x}_1)\chi_1(\vec{x}_0)\chi_2(\vec{x}_2)\chi_2(\vec{x}_0)\cdots$$

$$\cdots \chi_N(\vec{x}_N)\chi_N(\vec{x}_0)\} \exp -\frac{1}{2}\int \sum_{\alpha=1}^{M} \chi_\alpha(\vec{u})\, G^{-1}(\vec{u},\vec{w})\, \chi_\alpha(\vec{w})\, d^D u\, d^D w \qquad (52)$$

where during the integration we take $M > N$ and then evaluate the $M \to 0$ limit after the integral is done. The M-N integrals where components of χ_α don't appear in the braces give $(\eta)^{N-M}$ which as $M \to 0$ gives exactly the product of Green functions desired. Again all F dependence lies in the exponential factor, so the desired average over F is

$$\langle \phi(\vec{x}_1) \cdots \phi(\vec{x}_N) \rangle = \lim_{M \to 0} \int \prod_{\substack{\alpha=1 \\ \vec{u}}}^{M} d\chi_\alpha(\vec{u}) \left\{ \chi_1(\vec{x}_1)\chi_1(\vec{x}_0) \cdots \right.$$

$$\left. \cdots \chi_N(\vec{x}_N)\chi_N(\vec{x}_0) \right\} \exp - \int d^D u \, \mathcal{L}_{eff}(\chi_\alpha) \quad . \tag{53}$$

All of these quantities can be evaluated from the generating functional

$$Z[J_\beta] = \int \prod_{\substack{\alpha=1 \\ \vec{u}}}^{M} d\chi_\alpha(\vec{u}) \; \exp - \int d^D u \left[\mathcal{L}_{eff}(\chi_\alpha) - \sum_{\beta=1}^{M} J_\beta(\vec{u})\chi_\beta(\vec{u}) \right] , \tag{54}$$

by taking $M \to 0$ after computing the appropriate derivatives with respect to $J_\beta(\vec{x})$ at $J_\beta = 0$.

There is an alternative formulation of the generating functional for the present problem which is a bit less elegant, but is easier to work with. We again consider an auxiliary pseudostochastic field $\chi(\vec{x})$ which is gaussian, with zero mean and correlation function

$$G(\vec{x}_1, \vec{x}_0) = \langle \chi(\vec{x}_1)\chi(\vec{x}_0) \rangle_\chi \; . \tag{55}$$

For this field evaluate

$$\begin{aligned} \langle \chi(\vec{x}_1)\chi(\vec{x}_2)\chi(\vec{x}_0)^2 \rangle_\chi = {} & G(\vec{x}_1, \vec{x}_2)\, G(\vec{x}_0, \vec{x}_0) \\ & + 2!\, G(\vec{x}_1, \vec{x}_0)\, G(\vec{x}_2, \vec{x}_0) \; ; \end{aligned} \tag{56}$$

the last term is just 2! $\phi(\vec{x}_1)\,\phi(\vec{x}_2)$. Next compute

$$\langle \chi(\vec{x}_1)\chi(\vec{x}_2)\chi(\vec{x}_3)\chi(\vec{x}_0)^3\rangle_\chi = G(\vec{x}_1,\vec{x}_2)\,G(\vec{x}_3,\vec{x}_0)\,G(\vec{x}_0,\vec{x}_0) + \text{permutations} + 3!\,G(\vec{x}_1,\vec{x}_0)\,G(\vec{x}_2,\vec{x}_0)\,G(\vec{x}_3,\vec{x}_0). \qquad (57)$$

The last term is just 3! $\phi(\vec{x}_1)\phi(\vec{x}_2)\,\phi(\vec{x}_3)$. If we make a graphical representation of these averages with lines representing $G(\vec{x}_a,\vec{x}_b)$, then graphs with $G(\vec{x}_0,\vec{x}_0)$ or $G(\vec{x}_i,\vec{x}_j)$ $i,j \neq 0$ are disconnected. They can be separated into two parts without cutting any lines. The connected parts are what we want. More precisely

$$\phi(\vec{x}_1)\cdots\phi(\vec{x}_N) = \frac{1}{N!}\langle \chi(\vec{x}_1)\cdots\chi(\vec{x}_N)\,\chi(\vec{x}_0)^N\rangle_{\chi\ \text{connected}} \qquad (58)$$

$$= \frac{\eta}{N!}\int \prod_{\vec{x}} d\chi(\vec{x})\left\{\chi(\vec{x}_1)\cdots\chi(\vec{x}_N)\,\chi(\vec{x}_0)^N\right\} \exp -\tfrac{1}{2}\int \chi G^{-1}\chi \,\Big|_{\text{connected}} \qquad (59)$$

We actually wish, as usual, the average of this formula over the fluctuations in F. Consider for this the quantity

$$\frac{\bar{\eta}}{N!}\int d\chi(\vec{x})\,dF(\vec{u})\,P[F(\vec{u})]\,\left\{\chi(\vec{x}_1)\cdots\chi(\vec{x}_N)\,\chi(\vec{x}_0)^N\right\} \exp -\tfrac{1}{2}\int \chi G^{-1}\chi \,\Big|_{\text{connected}}, \qquad (60)$$

where

$$\bar{\eta}^{-1} = \int d\chi(\vec{x})\,dF(\vec{u})\,P[F(\vec{u})]\,\exp -\tfrac{1}{2}\int \chi G^{-1}\chi . \qquad (61)$$

This is the joint average over $\chi(\vec{x})$ and $F(\vec{x})$ of the quantity in braces. It generates, on expansion in the non-linearity in $\mathcal{L}_{eff} : -\log Z_F[\frac{1}{2}\chi^2]$ all the terms in the direct representation of $\langle G(\vec{x}_1,\vec{x}_0)\cdots G(\vec{x}_N,\vec{x}_0)\rangle_F$ plus terms like the one entering $G(\vec{x}_1,\vec{x}_0)$

$$\int \prod_{j=1}^{4} d^D z_j \; G_0(\vec{x}_1,\vec{z}_1)\,\Gamma(\vec{z}_1,\vec{z}_2)\,G(\vec{z}_2,\vec{z}_3)^2\,\Gamma(\vec{z}_3,\vec{z}_4)\,G_0(\vec{z}_1,\vec{z}_4)\,G_0(\vec{z}_4,\vec{x}_0) \qquad (63)$$

which is absent in the direct average over products of $(-G_0{}^{-1} + F)^{-1}$ expanded in F.

These extra terms represent corrections to the force-force correlation function $\Gamma(\vec{u},\vec{w})$ and since that function is uncorrected in the problem at hand, we may use the formula (60) to evaluate $\langle\phi(\vec{x}_1)\cdots\phi(\vec{x}_N)\rangle$, if we add the prescription that one must cast out all graphs that represent corrections or modifications to $\Gamma(\vec{x},\vec{y})$ (the wiggly line in Fig. 2). In quantum field theory, these insertions into the force-force correlation are essential and represent the dynamic aspect of the force field. Here the fluctuating field is not influenced by the passage of the wave through it, so we cast out all such effects in (60). With this rule implicit henceforth, we will use (60) in the next lecture.

To close this lecture I want to take another important example of propagation in a random medium: convection of a passive scalar $S(\vec{x},t)$ by a turbulent medium. The equations governing the motion are as usual

$$\frac{\partial}{\partial t} S(\vec{x},t) + \vec{v}\cdot\nabla S(\vec{x},t) = \varkappa_0 \nabla^2 S(\vec{x},t) + Q(\vec{x},t) \qquad (64)$$

where $v_j(\vec{x},t)$ is the velocity field of the turbulent flow, $\varkappa_0$ is some diffusion constant and $Q(\vec{x},t)$ a source for the scalar quantity $S(\vec{x},t)$.

Write this as

$$(\partial_t - \varkappa_0 \nabla^2 - F)\, S = Q \tag{65}$$

and call $\mathcal{G}_0^{-1}$ the operator

$$\mathcal{G}_0^{-1} = \partial_t - \varkappa_0 \nabla^2. \tag{66}$$

The operator $F = -\vec{v}\cdot\nabla$ fluctuates, and again we have a scalar quantity proceeding through a medium, the turbulent flow, without altering the properties of the medium.

The inverse operator $\mathcal{G}_0$ is

$$\mathcal{G}_0(\vec{x},t) = \int \frac{d^D q\, d\omega}{(2\pi)^{D+1}} \frac{e^{-i\omega t + i\vec{q}\cdot\vec{x}}}{-i\omega + \varkappa_0 \vec{q}^{\,2}} \tag{67}$$

$$= \theta(t) \int \frac{d^D q}{(2\pi)^D} e^{i\vec{q}\cdot\vec{x} - \varkappa_0 q^2 t}$$

$$= \theta(t)\left[\exp - \vec{x}^2/4\varkappa_0 t\right] / (4\pi\varkappa_0 t)^{D/2} \tag{68}$$

with $\theta(t) = 1,\ t > 0;\ \theta(t) = 0;\ t < 0.$

$S(\vec{x},t)$ is

$$S(\vec{x},t) = \int d^D y\, d\tau\, \mathcal{G}(\vec{x},t;\vec{y},\tau)\, Q(\vec{y},\tau), \tag{69}$$

with

$$\mathcal{G} = (\mathcal{G}_0^{-1} - F)^{-1} = \mathcal{G}_0 \frac{1}{1 - F\mathcal{G}_0} = \frac{1}{1 - \mathcal{G}_0 F}\mathcal{G}_0, \tag{70}$$

as in the previous case.

Following the example of scalar wave propagation we seek an auxiliary field, call it $\beta(\vec{x},t)$, which will allow us to express averages of $S(\vec{x},t)$ over the random function $F(\vec{x},t)$ via a generating functional. We would like β to be gaussian in some sense and have zero mean and correlation function

$$\mathcal{G} = \left(\mathcal{G}_0^{-1} - F\right)^{-1}$$

Since $\mathcal{G}_0^{-1}$ is linear in time derivatives, it is not possible to achieve our goal using only one field $\beta(\vec{x},t)$. Basically we are headed toward an action principle for the field β which will give

$$\left(\partial_t - \kappa_0 \nabla^2 - F\right) \beta(\vec{x},t) = 0 \qquad (71)$$

as the Euler-Lagrange equation of motion. An action principle is cooked up to describe conservative systems and here we have a dissipative system. We introduce, then, two fields: $\beta(\vec{x},t)$ and an anti-field $\bar{\beta}(\vec{x},t)$ which anti-diffuses so the net system is "conservative." The same issue appears in formulating a Lagrangian for the Schrödinger equation and in that case $\bar{\beta}$ is precisely the hermitian conjugate field. With real dissipation, $\bar{\beta}(\vec{x},t)$ has no completely compelling physical meaning. The idea of an anti-dissipating quantity is the best I've been able to do.

The lagrange density for β and $\bar{\beta}$ is

$$\mathcal{L}(\beta,\bar{\beta}) = \frac{1}{2}\,\bar{\beta}(\vec{x},t)\,\overleftrightarrow{\frac{\partial}{\partial t}}\,\beta(\vec{x},t) + \kappa_0 \nabla_j \bar{\beta}(\vec{x},t) \nabla_j \beta(\vec{x},t)$$

$$+ \bar{\beta}(\vec{x},t)\,\vec{v}(\vec{x},t)\cdot\nabla\beta(\vec{x},t)\,, \qquad (72)$$

with

$$A \overleftrightarrow{\frac{\partial}{\partial t}} B = A \frac{\partial B}{\partial t} - \frac{\partial A}{\partial t} B \, . \tag{73}$$

Next we introduce $\eta(\vec{x},t)$ and $\bar{\eta}(\vec{x},t)$ to define the generating functional for $\beta, \bar{\beta}$ correlation functions

$$\tilde{Z}[\eta,\bar{\eta}] = \int \prod_{\vec{x},t} d\beta(\vec{x},t) \prod_{\vec{z},\tau} d\bar{\beta}(\vec{z},\tau) \; e^{-\int d^D u \, d\rho \left[\mathcal{L}(\beta,\bar{\beta}) - \eta\beta - \bar{\eta}\bar{\beta} \right]} \, . \tag{74}$$

Again we can do the β and $\bar{\beta}$ integrals here, but what we seek is Z averages over the fluctuations in $\vec{v}(\vec{x},t)$.

This requires a slight extension of our discussion of probability distribution functionals to distributions of vector fields like $\vec{v}(\vec{x},t)$. It is easy to extend that discussion by considering the vector label simply to be a discrete label

$$v_\ell(\vec{x},t) \qquad \ell = 1, 2, \cdots D$$

in addition to the continuous space-time labels. A gaussian distribution of $v_\ell(\vec{x},t)$ with zero mean velocity has the form

$$P[v_\ell(\vec{x},t)] = \eta \exp -\frac{1}{2} \int d^D x \, dt \, d^D y \, d\tau \sum_{\ell,n=1}^{D} v_\ell(\vec{x},t) M_{\ell n}(\vec{x},t;\vec{y},\tau) v_n(\vec{y},\tau) \, , \tag{75}$$

and in this distribution

$$\langle v_\ell(\vec{x},t) \, v_n(\vec{y},\tau) \rangle = \Gamma_{\ell n}(\vec{x},t;\vec{y},\tau) \, , \tag{76}$$

where

$$\int d^Dz\, d\rho \sum_{k=1}^{D} M_{jk}(\vec{x},t;\vec{z},\rho)\,\Gamma_{kn}(\vec{z},\rho;\vec{y},\tau) = \delta_{jn}\,\delta^D(\vec{x}-\vec{y})\,\delta(t-\tau). \qquad (77)$$

The generating functional we desire is

$$Z[\eta,\bar{\eta}] = \int \prod_{\substack{\vec{x},t \\ j=1,\cdots D}} dv_j(\vec{x},t)\;P[v_j]\;\tilde{Z}[\eta,\bar{\eta}]. \qquad (78)$$

Since $\mathcal{L}$ is linear in $\vec{v}$, we once again encounter the characteristic functional for $P[v_j]$

$$Z_v[J_\ell] = \int \prod_{\substack{n=1 \\ \vec{x},t}}^{D} dv_n(\vec{x},t)\;P[v_j]\;\exp\int d^Du\,dt'\sum_{k=1}^{D} J_k(\vec{u},t')\,v_k(\vec{u},t') \qquad (79)$$

which we need at $J_n(\vec{u},t') = -\bar{\beta}(\vec{u},t')\,\nabla_n\,\beta(\vec{u},t')$.

If $P[v]$ is our good old gaussian, we can evaluate the integral over v_j to write

$$Z[\eta,\bar{\eta}] = \int \prod_{\vec{x},t} d\beta(\vec{x},t) \prod_{\vec{y},\tau} d\bar{\beta}(\vec{y},\tau)\; e^{-\text{Effective Action} - \int \beta\eta + \bar{\beta}\bar{\eta}}, \qquad (80)$$

where the effective action is

$$\int d^Du\,dt'\,\mathcal{L}_{eff}(\beta,\bar{\beta}) =$$

$$\int d^Du\,dt'\left[\frac{1}{2}\,\bar{\beta}\,\frac{\overleftrightarrow{\partial}}{\partial t'}\,\beta + \varkappa_0\,\nabla_j\,\bar{\beta}\,\nabla_j\,\beta\right]$$

$$-\frac{1}{2}\int d^Dx\,dt\;d^Dy\,d\tau\;\bar{\beta}(\vec{x},t)\nabla_\ell\,\beta(\vec{x},t)\,\Gamma_{\ell n}(\vec{x},t;\vec{y},\tau)\,\bar{\beta}(\vec{y},\tau)\nabla_n\,\beta(\vec{y},\tau) \qquad (81)$$

If the distribution of v_j is not gaussian, then in place of the last term in (81) will apear $-\log Z_v[-\bar{\beta}\nabla_\ell \beta]$.

Finally, to reconstruct the correlation functions of the passive scalar $S(\vec{x},t)$ we need

$$S(\vec{x},t) = \int d^D y\, d\tau \, \langle \beta(\vec{x},t)\,\bar{\beta}(\vec{y},\tau)\rangle_{\beta,\bar{\beta}}\, Q(\vec{y},\tau), \qquad (82)$$

which leads to

$$\langle S(\vec{x}_1,t_1)\cdots S(\vec{x}_N,t_N)\rangle_v =$$

$$\int \prod_{k=1}^{N} d^D y_k\, d\tau_k \prod_{\ell=1}^{N} Q(\vec{y}_\ell,\tau_\ell) \times$$

$$\times \langle \beta(\vec{x}_1,t_1)\cdots\beta(\vec{x}_N,t_N)\,\bar{\beta}(\vec{y}_1,t_1)\cdots\bar{\beta}(\vec{y}_N,t_N)\rangle_{\beta,\bar{\beta},v} . \qquad (83)$$

$Z[\eta,\bar{\eta}]$ gives us precisely the needed $\beta,\bar{\beta}$ correlation functions on appropriate differentiation with respect to η and $\bar{\eta}$ then evaluated at $\eta=\bar{\eta}=0$; for example,

$$\langle \beta(\vec{x},t)\,\bar{\beta}(\vec{y},\tau)\rangle_{\beta,\bar{\beta},v} = \left.\frac{\partial^2 Z[\eta,\bar{\eta}]}{\partial\eta(\vec{x},t)\,\partial\bar{\eta}(\vec{y},\tau)}\right|_{\eta=\bar{\eta}=0} \qquad (84)$$

Again Eqs. (48) and (81) or their variants when $\vec{v}$ or F are not gaussian are the essential result of this lecture.

They demonstrate precisely the nature of the non-linearity in the dynamics of $\chi(\vec{x})$ or $\beta(\vec{x},t)$ and $\bar{\beta}(\vec{x},t)$ induced by the fluctuations of the medium—either sound speed fluctuations or turbulent velocity fluctuations. Furthermore, it gives an integral representation for the correlation functions needed to construct the averages over the acoustic pressure $\langle\phi(\vec{x}_1)\cdots\phi(\vec{x}_N)\rangle$ or the passive scalar $\langle S(\vec{x}_1,t_1)\cdots S(\vec{x}_N,t_N)\rangle$.

There are two more topics of some general interest which logically should be treated here in the fashion we have indicated: turbulent fluid flow and motion of a field-wave motion or dissipation in the presence of a random surface. We'll begin the second lecture with the first of these and relegate the second to an addendum—it will be discussed only if time permits.

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LECTURE 2 Turbulent Fluid Flow; Beginnings of Renormalization Group Analysis

In the examples we discussed in the previous lecture we had a scalar proceeding through a prescribed random medium without interacting back on the medium. The non-linearity of the resulting wave propagation or diffusion was <u>induced</u> by the fluctuations of the medium. Our next topic is turbulent fluid flow. It differs in an important sense from the previous examples because the non-linearity comes in the formulation of the theory while the stochasticity arises from some random initial conditions on the dynamical variables $v_j(\vec{x},t)$ - the velocity field. These random initial conditions can be generated by some given random external forces as we will explain below. The physical issue is the transport of these inital conditions to a distribution functional $P\left[v_j(x,t)\right]$ at a later time by the non-linear dynamical equations—the Navier-Stokes equations. As our first topic in this lecture I will cast this problem into the same form as we cast the scalar motion through a random medium in the previous lecture; namely, a generating functional for correlation functions of the stochastic velocity field will be derived. To focus attention on the issues at hand, I will assume that the fluid is incompressible and that the flow is homogeneous and isotropic. Both the stationary and non-stationary cases will be treated. More general flows can be studied with the methods to be indicated; the idealized example discussed will illustrate all of the essential points without adding to the notational burden already imposed by the basic structure of the equations.

We turn then to the Navier-Stokes equations

$$\frac{\partial v_j}{\partial t} + v_\ell \nabla_\ell v_j = -\frac{1}{\rho} \nabla_j p + \nu_0 \nabla^2 v_j + F_j \qquad (85)$$

where the density ρ is constant, p is the pressure, ν_0, the kinematic viscosity, and f_j is an external body force. Since ρ is constant, the velocity field is divergenceless and this allows us to eliminate the pressure in the usual fashion. Take the divergence of (85) to find

$$\frac{1}{\rho} \nabla^2 p = \nabla_j F_j - \nabla_\ell \nabla_j (v_\ell v_j). \qquad (86)$$

The operator $1/\nabla^2$ will mean

$$\frac{1}{\nabla^2} f(\vec{x}) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\vec{k}\cdot\vec{x}}}{-\vec{k}^2 + i\varepsilon} \hat{f}(\vec{k}) \qquad (87)$$

$$= -\frac{1}{4\pi} \int d^3y \frac{1}{|\vec{x}-\vec{y}|} f(\vec{y}), \text{ at } D=3, \qquad (88)$$

where $\hat{f}(\vec{k})$ is the fourier transform of $f(\vec{x})$. So

$$-\frac{1}{\rho} \nabla_j p = -\frac{1}{\nabla^2} \nabla_j \nabla_n F_n + \frac{\nabla_j \nabla_n}{\nabla^2} \nabla_\ell (v_n v_\ell), \qquad (89)$$

and

$$\frac{\partial v_j}{\partial t} + \Delta_{jn} \nabla_\ell (v_n v_\ell) = \nu_0 \nabla^2 v_j + \Delta_{jn} F_n, \qquad (90)$$

with

$$\Delta_{jn} = \delta_{jn} - \nabla_j \nabla_n / \nabla^2 \;) \qquad (91)$$

the projection operator on divergenceless vectors. Henceforth we'll take the external force divergence free since only that part of it enters the equation of motion.

Imagine that the force $F_j(\vec{x},t)$ is turned on at $t=0$ before which time $v_j(\vec{x},t) = 0$ and then turned off at some time $t=T$. If T is short, then as first argued by Saffman (1967) we have

$$v_j(\vec{x},T) = \int_0^T dt\, F_j(\vec{x},t) . \qquad (92)$$

We want $v_j(\vec{x},T)$ to be our random initial condition, so we take F_j to be a random forcing or mixing of the fluid with statistics the same as we desire for $v_j(\vec{x},T)$. For $t>T$, we can now ask how the fluid develops via (90).

Since we have a viscous medium, the velocity will decay for $t>T$ since energy is no longer being put in. As shown by Monin and Yaglom, for isotropic flow the only source of energy input to overcome the viscous dissipation is an external stirring force. The present discussion is then adequate for studying the development of non-stationary turbulence.

Stationary turbulence is conceptually simpler since there is an additional symmetry; namely, time translation invariance. To achieve a stationary flow we need to force the fluid continuously, so we have to let F_j, the random external force, be on forever. For the rest of this lecture we will focus on this kind of mixing.

There is no need of an auxiliary field now since it is the correlation functions of $v_j(\vec{x},t)$ that we seek directly. We proceed by constructing a lagrangian density yielding (90) as the Euler-Lagrange equation. Because of the dissipation we require an anti-velocity field $\bar{v}_j(\vec{x},t)$ for this construction. We are lead to

$$\mathcal{L}(v_j,\bar{v}_j) = \frac{1}{2}\,\bar{v}_j \overleftrightarrow{\frac{\partial}{\partial t}} v_j + \nu_0 \nabla_n \bar{v}_j \nabla_n v_j - \frac{1}{2}\Big(\big(\Delta_{jn}\nabla_\ell + \Delta_{j\ell}\nabla_n\big)\bar{v}_j\Big) v_\ell v_n - F_j \bar{v}_j \;. \qquad (93)$$

A generating functional for correlation functions like

$$\langle v_{\ell_1}(\vec{x}_1,t_1)\cdots v_{\ell_N}(\vec{x}_N,t_N)\,\bar{v}_{k_1}(\vec{y}_1,\tau_1)\cdots \bar{v}_{k_m}(\vec{y}_m,\tau_m)\rangle, \qquad (94)$$

where $\langle\;\rangle$ means average over the probability distribution functional of F_j, is given by

$$Z[\eta_j,\bar{\eta}_j] = \int \prod_{\substack{n=1\\ \vec{x},t}}^{D} dv_n(\vec{x},t) \prod_{\substack{k=1\\ \vec{y},\tau}}^{D} d\bar{v}_k(\vec{y},\tau) \prod_{\substack{j=1\\ \vec{z},\rho}}^{D} dF_j(\vec{z},\rho)\,\times$$

$$\times\, P[F_j(\vec{z},\rho)]\; \exp - \int d^D u\, dt' \left[\mathcal{L}(v_j,\bar{v}_j) - \eta_j v_j - \bar{\eta}_\ell \bar{v}_\ell\right]. \qquad (95)$$

As ever, the lagrangian density is linear in F_j so we recognize

$$Z_F[J_n] = \int \prod_{\substack{j=1 \\ \vec{z},\rho}}^{D} dF(\vec{z},\rho)\ P[F_j]\, exp \int d^D u\, dt\ F_j(\vec{u},t)\, J_j(\vec{u},t) \tag{96}$$

evaluated at $J_n(\vec{x},t) = \bar{v}_n(\vec{x},t)$. The effective lagrangian density for turbulent flow is

$$\mathcal{L}_{eff}(v_j,\bar{v}_j) = \frac{1}{2}\bar{v}_j \overleftrightarrow{\frac{\partial}{\partial t}} v_j + \nu_0 \nabla_n \bar{v}_j \nabla_n v_j$$

$$-\frac{1}{2}\left(\left(\Delta_{jn}\nabla_\ell + \Delta_{j\ell}\nabla_n\right)\bar{v}_j\right) v_\ell v_n - \log Z_F[\bar{v}_n]. \tag{97}$$

Suppose the mixing force is gaussian. It must have zero mean since we have assumed isotropy of the flow. So we write

$$P[F_j] = exp -\frac{1}{2}\int d^D x\, dt\, d^D y\, d\tau\ F_j(\vec{x},t)\, M_{j\ell}(\vec{x},t;\vec{y},\tau)\, F_\ell(\vec{y},\tau), \tag{98}$$

and

$$\langle F_j(\vec{x},t)\, F_\ell(\vec{y},\tau)\rangle_F = \tilde{\Gamma}_{j\ell}(\vec{x}-\vec{y},t-\tau), \tag{99}$$

where space and time differences appear because of homogeneity and stationarity. Since F_j is divergenceless and we have isotropy, $\tilde{\Gamma}_{j\ell}$ has the form

$$\tilde{\Gamma}_{j\ell}(\vec{x}-\vec{y},t-\tau) = \Delta_{j\ell}\ \hat{\Gamma}\left((\vec{x}-\vec{y})^2, t-\tau\right). \tag{100}$$

For the gaussian the generating functional Z_F is

$$\log Z_F[\bar{v}_n] = \frac{1}{2}\bar{v}_n \int d^D y\, d\tau\, \hat{\Gamma}((\vec{x}-\vec{y})^2, t-\tau)\, \bar{v}_n(\vec{y},\tau)\,, \quad (101)$$

remembering $\nabla_n \bar{v}_n = 0$.

Putting together these results for a gaussian mixing force, or equivalently in the non-stationary situation a gaussian set of initial conditions, we can write the generating functional for v, $\bar{v}$ correlation functions

$$Z[\eta_j, \bar{\eta}_j] = \int dv_j\, d\bar{v}_j\, \exp - \int [\mathcal{L}_{eff} - \eta_j v_j - \bar{\eta}_j \bar{v}_j]\,, \quad (102)$$

where

$$\mathcal{L}_{eff} = \frac{1}{2}\bar{v}_j \overleftrightarrow{\frac{\partial}{\partial t}} v_j + \nu_0 \nabla_n \bar{v}_j \nabla_n v_j - \frac{1}{2}\bar{v}_n \hat{\Gamma} \bar{v}_n$$

$$- \frac{1}{2}\left((\Delta_{jn}\nabla_\ell + \Delta_{j\ell}\nabla_n)\bar{v}_j \right) v_n v_\ell \,. \quad (103)$$

We see directly that even though the forcing was gaussian, the distribution of v_j is not. The cause, clearly, is the non-linearity of the Navier-Stokes equation which has surfaced as the last term in (103). One could do either the v_j or the $\bar{v}_j$ integral in (103) because $\mathcal{L}_{eff}$ is at most quadratic in either; however, we will proceed by another route later.

Having cast the problem of homogeneous, isotropic, stationary turbulence into the same general form as our description of propagation or diffusion of a scalar in a random medium, we return to those as they are somewhat simpler to deal with in the next stage of our development which is the discussion of the renormalization group.

We are actually going to discuss a slightly more simplified problem so we may focus on the issues. In the effective lagrangian for propagation of scalars in a gaussian random medium we encounter the correlation function $\Gamma(\vec{x}-\vec{y})$ of the fluctuations. This contains information on the magnitude of the fluctuations as well as the spatial scales over which the correlation is significant. I will set these scales to zero and write

$$\Gamma(\vec{x}-\vec{y}) = -\frac{\lambda_0}{3}\delta^D(\vec{x}-\vec{y}). \qquad (104)$$

We may restore the scales later, if we wish.

The effective problem we want to study is then

$$Z[J] = \int \prod_{\vec{x}} d\chi(\vec{x})\, e^{-\int d^D u \left[\frac{(\nabla\chi)^2}{2} - k_0^2\frac{\chi^2}{2} + \frac{\lambda_0}{4!}\chi(\vec{u})^4 - J\chi\right]} . \qquad (105)$$

We will proceed by considering the perturbation series in λ_0 for various correlation functions. Immediately we will be led to the idea that the actual expansion parameter is not the constant λ_0 but is a function of $\vec{x}$ or wave number—the Fourier conjugate variable to $\vec{x}$. We will then define a dimensionless expansion parameter which depends on an arbitrary scale in wave number space. As we vary this scale we will be able to determine where the effective dimensionless expansion parameter is small or large and thus determine how good the expansion itself will be.

Let's begin with the function

$$H^{(2)}(\vec{x}-\vec{y}) = \langle \chi(\vec{x})\chi(\vec{y})\rangle = \int \prod_{\vec{u}} d\chi(\vec{u}) \;\; \chi(\vec{x})\chi(\vec{y})\, e^{-\int d^D w\left[\frac{1}{2}(\nabla\chi)^2 - \frac{1}{2}k_0^2\chi^2 + \frac{\lambda_0}{4!}\chi^4\right]} \qquad (106)$$

$$= H_0(\vec{x}-\vec{y}) - \frac{\lambda_0}{2}\int d^D z \;\; H_0(\vec{x}-\vec{z})H_0(\vec{z}-\vec{y}) \;\; H_0(0) +$$

$$+ \left(\frac{\lambda_0}{2}\right)^2 \int d^D z \;\; H_0(\vec{x}-\vec{z})H_0(\vec{z}-\vec{y}) \;\; (H_0(0))^2 +$$

$$+ \left(\frac{\lambda_0}{2}\right)^2 \int d^D z \, d^D w \; H_0 \; (\vec{x}-\vec{z})H_0(\vec{z}-\vec{w})\, H_0(\vec{w}-\vec{y}) \;\; (H_0(0))^2 +$$

$$+ \frac{\lambda_0^2}{3!} \int d^D z \;\; d^D w \;\; H_0(\vec{x}-\vec{z})\left(H_0(\vec{z}-\vec{w})\right)^3 H_0 \; (\vec{w}-\vec{y})$$

$$+ \; O(\lambda_0^3) \qquad (107)$$

where

$$H_0(\vec{x}) = \int \frac{d^D p}{(2\pi)^D} \;\; \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2 - k_0^2 - i\varepsilon} \;\; = \; - G_0(\vec{x}) \qquad (108)$$

as we would expect from Eq. (42). The terms of (107) are shown in Fig. 4. They are just the graphs in Fig. 2 with the wiggly Γ line reduced to a point.

It is more transparent to write the series in wave number space where

$$H^{(2)}(p^2) = \frac{1}{p^2 - k_0^2 - i\varepsilon} - \frac{\lambda_0}{2}\int \frac{d^D k}{(2\pi)^D} \;\; \frac{1}{k^2 - k_0^2 - i\varepsilon} \; \left(\frac{1}{p^2 - k_0^2 - i\varepsilon}\right)^2 +$$

$$+\left(\frac{\lambda_0}{2}\right)^2\left[\left(\frac{1}{p^2-k_0^2-i\varepsilon}\right)^2+\left(\frac{1}{p^2-k_0^2-i\varepsilon}\right)^3\right]\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2-k_0^2-i\varepsilon}\;+$$

$$+\frac{\lambda_0^2}{3!}\left(\frac{1}{p^2-k_0^2-i\varepsilon}\right)^2\int\frac{d^Dk_1d^Dk_2d^Dk_3}{(2\pi)^{3D}}\,(2\pi)^D\delta^D(k_1+k_2+k_3-p)\prod_{j=1}^{3}\frac{1}{k_j^2-k_0^2-i\varepsilon}$$

$$+\;O(\lambda_0^3)\quad . \tag{109}$$

Examination of these graphs and of higher order graphs shows that $H^{(2)}(p^2)$ can be written as

$$H^{(2)}(p^2)^{-1}=p^2-k_0^2-\Sigma(p^2)-i\varepsilon\equiv\Gamma^{(2)}(p^2), \tag{110}$$

where to order λ_0^2

$$\Sigma(p^2)=-\frac{\lambda_0}{2}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2-k_0^2-i\varepsilon}+\left(\frac{\lambda_0}{2}\right)^2\left(\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2-k_0^2-i\varepsilon}\right)^2$$
$$+\frac{\lambda_0^2}{3!}\int\frac{d^Dk_1\,d^Dk_2\,d^Dk_3}{(2\pi)^{2D}}\,\delta^D(k_1+k_2+k_3-p)\prod_{j=1}^{3}\frac{1}{k_j^2-k_0^2-i\varepsilon}\quad . \tag{111}$$

The graphs contributing to Σ are shown in Fig. 5. Σ is often called the <u>mass operator</u> or <u>self energy</u>. It reflects the real dynamics entering into the calculation of $H(g^2)$. In the present context a good name for $\Sigma(p^2)$ would be the <u>self interaction contribution</u> to the wave propagator. It contains the effects of fluctuations of the medium and the manner in which they cause the waves to interact back on themselves.

$\Sigma(p^2)$ represents the full modification to the original wave equation since $\left(p^2-k_0^2-\Sigma(p^2)\right)\langle\phi(p^2)\rangle = 1$ could be considered a renormalized or, better, self corrected wave equation. The multiple poles at $p^2 = k_0{}^2$ are artifical and misleading. Frisch (1968) shows how they lead to so-called "secular" behavior.

In graphical language what we have done in isolating Σ is to "chop off the external legs" - i.e., two powers of $(p^2-k_0{}^2)^{-1}$ and also select those graphs which cannot be cut into two separate parts by cutting one line. This produces the so-called amputated one line irreducible vertex functions. One can give a formal definition of these irreducible functions. We really only need the definition for

$$H^{(4)}(\vec{x}_1, \cdots \vec{x}_4) = \langle \chi(\vec{x}_1) \cdots \chi(\vec{x}_4) \rangle \tag{112}$$

and that is done in pictures in Fig. 6. We will call $\Gamma^{(4)}$ the amputated one line irreducible part of $H^{(4)}$. It has the external legs of $H^{(4)}$ amputated and only keeps graphs which are connected: the terms $H^{(2)}H^{(2)}$ appearing in $H^{(4)}$ are not in $\Gamma^{(4)}$. Speaking loosely, $\Gamma^{(4)}$ represents the guts of the <u>interaction</u> between the waves propagating in the random medium.

The lowest order contributions to $\Gamma^{(4)}$ are shown in Fig. 7. Algebraically they are

$$\Gamma^{(4)}(\vec{p}_1, \cdots, \vec{p}_4) = -\lambda_0 + \frac{\lambda_0^2}{2}\left[\alpha(\vec{p}_1+\vec{p}_2) + \alpha(\vec{p}_1+\vec{p}_3) + \alpha(\vec{p}_1+\vec{p}_4)\right] + $$

$$+ O(\lambda_0^3), \tag{113}$$

where

$$\alpha(\vec{p}) = \int \frac{d^D k_1 d^D k_2}{(2\pi)^D} \delta^D(\vec{k}_1 - \vec{k}_2 + \vec{p}) \prod_{j=1}^{2} \frac{1}{k_j^2 - k_0^2 - i\varepsilon} . \qquad (114)$$

This series tells us how strongly waves in the medium interact with each other. The series in $\Gamma^{(2)}$ tells us how strongly waves in the medium interact with themselves. $\Gamma^{(4)}$ defines an interaction strength that is wave number dependent. It is useful to define a renormalized interaction parameter at some arbitrary, symmetric point in wave number space, say where all $\vec{p}_i^{\,2} = \mu^2$ in $\Gamma^{(4)}(\vec{p}_1, \cdots \vec{p}_4)$; $\mu^2 > 0$. Calling this renormalized interaction parameter λ , we have

$$\Gamma^{(4)}(\vec{p}_i)\Big|_{\vec{p}_i^{\,2} = \mu^2} \propto -\lambda(\mu^2) \qquad (115)$$

Clearly the value of λ will depend on μ^2 , which is quite arbitrary. So λ cannot have any physical significance in itself. It is an infinite series in λ_0 and could be used as an expansion parameter as well as λ_0 .

Let us look back now at the series for $\Gamma^{(2)}(p^2) = H^{(2)}(p^2)^{-1} = p^2 - k_0^2 - \Sigma(p^2)$. Because of the interaction, the coefficient of p^2 in $\Gamma^{(2)}$ and the constant term in $\Gamma^{(2)}$ are changed. Expanding $\Sigma(p^2) = \Sigma(0) + p^2 \Sigma'(0) + \cdots$, we see

$$\Gamma^{(2)}(p^2) = p^2\left(1 + \Sigma'(0)\right) - k_0^2\left(1 + \Sigma(0)\right) + \cdots . \qquad (116)$$

So just as λ_0 was renormalized by the interaction, so are p^2 - which reflects the basic $\frac{1}{2}(\nabla\chi)^2$ term in the effective lagrangian—and k_0^2—which reflects the $-k_0^2\chi^2/2$ term in the lagrangian.

All of the basic parameters in the lagrangian are rescaled by the interaction. We can "undo" this rescaling by changing over to scaled fields via

$$\chi_R(\vec{x}) = Z^{1/2}\,\chi(\vec{x}) \qquad (117)$$

so

$$H_R^{(2)}(\vec{x}-\vec{y}) = Z\,H^{(2)}(\vec{x}-\vec{y}), \qquad (118)$$

and

$$\Gamma_R^{(2)}(p^2) = Z^{-1}\,\Gamma^{(2)}(p^2). \qquad (119)$$

Also we can define a rescaled $k_0^{\,2}$ by

$$k_R^2 = Z_0\,k_0^2 \qquad (120)$$

To determine Z and Z_0 we ask that

$$\Gamma_R^{(2)}(p^2)\Big|_{p^2=\mu^2} = Z^{-1}\left(\mu^2 - k_R^2\right), \qquad (121)$$

$$= Z^{-1}\left(\mu^2 - k_0^2 - \Sigma(p^2=\mu^2)\right), \qquad (122)$$

so

$$Z_0 = 1 + \frac{1}{k_0^2}\,\Sigma(\mu^2), \qquad (123)$$

and also

$$\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p^2)\Big|_{p^2=\mu^2} = 1 = Z^{-1}\,\frac{\partial}{\partial p^2}\Gamma^{(2)}(p^2)\Big|_{p^2=\mu^2}. \qquad (124)$$

This gives

$$Z = 1 - \left.\frac{\partial \Sigma}{\partial p^2}\right|_{p^2=\mu^2} . \qquad (125)$$

In all this, the parameter μ^2 where I choose to define the dimensionless scaling coefficients Z and Z_0 is the same as in the redefinition of λ from $\Gamma^{(4)}$. That is a convenience; it is not necessary to choose these <u>renormalization</u> <u>points</u> - which are arbitrary after all—the same,

Why have we done all this? Well, as one can see from the series for $\Gamma^{(2)}$ or $\Gamma^{(4)}$ the size of the correction terms to $-\lambda_0$ or to $p^2-k_0^2$ are dependent on wave number. Those corrections are the direct representation of the non-linearity induced by the medium. We want to examine how important that non-linearity is in any given regime of k space. The deviations of Z, Z_0, or λ/λ_0 from unity are also a measure of the importance of the non-linearity.

In addition, we will find that the series in λ_0 may be quite a divergent one,since the dimensionless version of λ_0 can become quite large in certain regions of wave number space. The parameter λ is an infinite series in λ_0 and may itself serve as a better expansion parameter than λ_0. This mapping from λ_0 to λ which improves the convergence of the series expansion of $\Gamma^{(2)}$ or $\Gamma^{(4)}$, or whatever, is one of the key goals of the present analysis.

Just an aside here. At D=4 dimensions, most of the integrals we have written down in the series for $\Gamma^{(2)}$ or $\Gamma^{(4)}$

are divergent. To define the theory then, one needs a renormalization prescription like the one indicated. In the present case we do have certain integrals to be careful about, but the renormalization is not necessary. It is a tool for exploring the importance of the non-linearity in any given realm of k space. This attribute will be clear soon.

We need one more useful set of observations. The parameter λ_0 (or λ) is not dimensionless, so it doesn't provide a "true" measure of the size of the terms in perturbation theory. To identify a dimensionless parameter of expansion we need to indulge in a bit of dimensional analysis.

Return to the action

$$\int d^D x \left[\frac{1}{2} (\nabla \chi)^2 - \frac{1}{2} k_0^2 \chi^2 + \frac{\lambda_0}{4!} \chi^4 \right].$$

It is dimensionless. Let [quantity] represent the dimensions of a quantity in powers of wave number k. So

$$[x] = k^{-1} . \tag{126}$$

Then from the action we learn

$$[\chi(\vec{x})] = k^{(D-2)/2} , \tag{127}$$

$$[k_0] = k ,$$

and

$$[\lambda_0] = k^{4-D} . \tag{128}$$

We could construct $\lambda_0 k_0^{D-4}$ and call it our dimensionless parameter, but then it would not help us probe the size of the non-linearity in k space. Instead we form the two dimensionless quantities

$$g_0 = \lambda_0 \mu^{D-4} \tag{130}$$

and

$$r_0 = k_0^2 / \mu^2 , \tag{131}$$

or in terms of renormalized parameters

$$g = \lambda \mu^{D-4} \tag{132}$$

and

$$r = k_R^2 / \mu^2 . \tag{133}$$

We are now ready to explore the consequences of the arbitrariness of μ and to see how it helps us explore different realms of k space. This is the guts of the <u>renormalization group</u> method. To draw even closer attention to the method, I will butcher the already butchered problem by setting $k_0 = 0$. This gives us only one dimensionless parameter to deal with. $k_0 \neq 0$ can be restored, and I will indicate how— after the smoke clears.

Suppose we have evaluated as many terms in the series in λ_0 for, say, $\Gamma^{(2)}$ as we are able. We will have constructed a function $\Gamma^{(2)}(p^2, \lambda_0)$. If we go over to λ and $\Gamma_R^{(2)}$, we will have a re-organized series, now in λ and not in λ_o. These functions are related at every p^2 by

$$Z(g)\,\Gamma_R^{(2)}(p^2, g, \mu^2) = \Gamma^{(2)}(p^2, \lambda_o). \qquad (134)$$

The function $\Gamma^{(2)}(p^2, \lambda_o)$ never heard about the normalization point μ^2 where the new expansion parameter λ was defined, so it must be true that

$$\mu^2 \frac{\partial}{\partial\mu^2}\Gamma^{(2)}(p^2, \lambda_o) = 0 \quad . \qquad (135)$$

Since, in principle, $\Gamma^{(2)}$ is the full function determining $\langle\phi(\vec{x})\rangle$:

$$\langle\phi(\vec{x})\rangle = \int \frac{d^D p}{(2\pi)^D}\,\frac{e^{i\vec{p}\cdot\vec{x}}}{\Gamma^{(2)}(p^2, \lambda_o) - i\varepsilon} \quad , \qquad (136)$$

all relevant physical information is in it. Its independence of μ expresses the arbitrariness of μ and emphasizes unphysical nature of μ. The utility of μ is about to emerge.

From the chain rule and (134) we derive from (135)

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + A(g)\frac{\partial}{\partial g} + C(g)\right]\Gamma_R^{(2)}(p^2, g, \mu^2) = 0 \; , \qquad (137)$$

where

$$A(g) = \mu^2 \frac{\partial}{\partial\mu^2}\, g \,\bigg|_{\lambda_o \text{ fixed}} \; , \qquad (138)$$

and

$$C(g) = \mu^2 \frac{\partial}{\partial\mu^2} \log Z(g) \bigg|_{\lambda_o \text{ fixed}} \quad . \qquad (139)$$

This tells us how we must vary g and the scale of χ as μ is varied, so no change occurs in the physics. This is the renormalization group equation. It is an exact partial differential equation which $\Gamma_R^{(2)}$ must satisfy. If we know the coefficients A(g) and C(g) even as a series in g, the solution of the differential equation will give us a non-perturbative solution for $\Gamma_R^{(2)}$ and hence for $\Gamma^{(2)}$ via (134). If we knew either $\Gamma_R^{(2)}$ or $\Gamma^{(2)}$ exactly, all this would be merely a check on our arithmetic—and not even an enjoyable one. Let me repeat that an approximation to A(g) and C(g) will give an improved version of $\Gamma^{(2)}$ in a manner to be demonstrated explicitly soon.

Let us return for a moment to a bit of dimensional analysis. Since $\Gamma_R^{(2)}(p^2, g, \mu^2)$ has dimensions k^2 it can be written

$$\Gamma_R^{(2)}(p^2, g, \mu^2) = p^2 \; \psi\left(p^2/\mu^2 \;, g\right) \tag{140}$$

So

$$\Gamma_R^{(2)}(\xi p^2, g, \mu^2) = \xi p^2 \; \psi\left(p^2/(\mu^2/\xi) \;, g\right) \tag{141}$$

$$= \xi\, \Gamma_R^{(2)}\left(p^2, g, \mu^2/\xi\right) . \tag{142}$$

This means

$$\xi \frac{\partial}{\partial \xi} \Gamma_R^{(2)}(\xi p^2, g, \mu^2) = \left(1 - \mu^2 \frac{\partial}{\partial \mu^2}\right) \Gamma_R^{(2)}(\xi p^2, g, \mu^2), \tag{143}$$

and the renormalization group equation reads

$$\left[\xi \frac{\partial}{\partial \xi} - A(g) \frac{\partial}{\partial g} - (C(g)-1) \right] \Gamma_R^{(2)}(\xi p^2, g, \mu^2) = 0. \qquad (144)$$

To solve this we use the method of characteristics which introduces $\hat{g}(\tau)$ satisfying

$$\frac{d\hat{g}(\tau)}{d\tau} = - A(\hat{g}(\tau)) \; ; \; \hat{g}(0) = g \; ; \; \tau = \log \xi \quad . \qquad (145)$$

This allows us to case (144) into

$$\frac{d}{dt} \Gamma_R^{(2)}(e^t p^2, \hat{g}(t), \mu^2) = \left[C(\hat{g}(t)) - 1 \right] \Gamma_R^{(2)}(e^t p^2, \hat{g}(t), \mu^2). \qquad (146)$$

Utilizing the absence of explicit τ dependence on the right hand side of (145), we can cast the solution to the renormalization group equation into

$$\Gamma_R^{(2)}(\xi p^2, g, \mu^2) = \Gamma_R^{(2)}(p^2, \hat{g}(-\log \xi), \mu^2) \times$$

$$\times \exp \int_{-\log \xi}^{0} d\tau' \left[C(\hat{g}(\tau')) - 1 \right] \qquad (147)$$

This is a traditional moment of rejoicing in the use of the renormalization group to study the dynamics of non-linear stochastic systems. It shows explicitly how the function $\Gamma^{(2)}$ at wave number $\sqrt{\xi}\vec{p}$ depends on a dimensionless expansion parameter $\hat{g}(-\log \xi)$ which clearly varies as the wave number $\sqrt{\xi}\vec{p}$ varies. If $\hat{g}(-\log \xi)$ is ever small, then a perturbation expansion of $\Gamma_R^{(2)}$ is going to yield an excellent approximation to $\Gamma_R^{(2)}$. (It is true we'll need Z(g) also, but that will come shortly.) The only information we need is the function A(g), for then $\hat{g}(-\log\xi)$ is determined, (145).

We can determine A(g) and C(g) or equivalently $g(\mu^2)$ and Z(g) by perturbation theory. Other techniques rely on a saddlepoint method for doing the appropriate functional integral. The information we need is in (114) and the definition of λ via

$$\Gamma_R^{(4)}(\vec{p}_i)\Big|_{\vec{p}_i^{\,2}=\mu^2} = -Z^{-2}\lambda = Z^{-2}\,\Gamma^{(4)}(\vec{p}_i)\Big|_{\vec{p}_i^{\,2}=\mu^2}\;, \tag{148}$$

and (125), the definition of Z. Because the p^2 dependence in $\Sigma(p^2)$ is $O(\lambda_0^2)$ we have,

$$Z = 1 + O(\lambda_0^2) \tag{149}$$

and to $O(\lambda_0^2)$ it does not affect the λ, λ_0 relation which is

$$\lambda = \lambda_0 - \frac{\lambda_0^2}{2}\left[\alpha(\vec{p}_1+\vec{p}_2) + \alpha(\vec{p}_1+\vec{p}_3) + \alpha(\vec{p}_1+\vec{p}_4)\right]\Big|_{\vec{p}_i^{\,2}=\mu^2}\;, \tag{150}$$

$$= \lambda_0 - \frac{3\lambda_0^2}{2}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2-i\varepsilon}\,\frac{1}{(\vec{k}+\vec{p})^2-i\varepsilon}\Big|_{p^2=\frac{4\mu^2}{3}} \tag{151}$$

$$= \lambda_0 - \frac{3\lambda_0^2}{2}\, I\left(p^2 = 4\mu^2/3\right).$$

To do this integral we use the identity

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}\;, \tag{152}$$

to cast it into

$$(2\pi)^D\, I(p^2) = \int_0^1 dx \int \frac{d^D k}{[k^2 + 2\vec{p}\cdot\vec{k}\,x + xp^2 - i\varepsilon]^2} \tag{154}$$

$$= \int_0^1 dx \int \frac{d^D q}{[q^2 + p^2 x(1-x) - i\varepsilon]^2} \tag{155}$$

after changing variables to $\vec{q} = \vec{k} + x\vec{p}$.

In D dimensions we can evaluate this using the result

$$\int \frac{d^D k}{[k^2 + Q]^N} = \pi^{D/2}\, Q^{\frac{D}{2}-N}\, \Gamma(N - D/2) \big/ \Gamma(N) \quad , \tag{156}$$

so

$$I\left(p^2 = \tfrac{4\mu^2}{3}\right) = (\mu^2)^{\frac{D}{2}-2} \frac{\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}} \left(\frac{4}{3}\right)^{\frac{D}{2}-2} \frac{\Gamma(\frac{D}{2}-1)^2}{\Gamma(D-2)} \quad , \tag{157}$$

and

$$\lambda = \lambda_0 - \frac{3}{2} \frac{\lambda_0^2 (\mu^2)^{\frac{D}{2}-2}}{(4\pi)^{D/2}} K_D \; . \tag{158}$$

At $D=3$, $K_3 = \sqrt{3}\,\pi^{3/2}/2$. K_D is positive for $2<D<4$. From all this we can determine $A(g)$

$$A(g) = \mu^2 \frac{\partial}{\partial\mu^2} g \Big|_{\lambda_0 \text{ fixed}} \tag{159}$$

$$= \mu^2 \frac{\partial}{\partial\mu^2} \lambda(\mu^2)^{(D-4)/2} \Big|_{\lambda_0 \text{ fixed}} \tag{160}$$

$$= -\frac{(4-D)}{2} g + 3\left(1 - \frac{D}{4}\right) K_D \frac{g^2}{(4\pi)^{D/2}} \quad , \tag{161}$$

or

$$A(g) = -\frac{(4-D)}{2}\,g + a\,g^2 . \tag{162}$$

At $D=3$, $a=\frac{9\sqrt{3}}{64} > 0$.

This result enables us to solve for $\tilde{g}(\tau)$

$$\frac{d\hat{g}(\tau)}{d\tau} = \frac{4-D}{2}\,\hat{g}(\tau) - a\,\hat{g}(\tau)^2 , \tag{163}$$

$$\hat{g}(\tau) = \frac{e^{(4-D)\tau/2}\, g}{1 + g\left(\frac{2a}{4-D}\right)\left(e^{(4-D)\tau/2} - 1\right)} , \tag{164}$$

or

$$\hat{g}(-\log\xi) = \frac{g\,\xi^{-(4-D)/2}}{1 + g\left(\frac{2a}{4-D}\right)\left(\xi^{-(4-D)/2} - 1\right)} . \tag{165}$$

As $\xi \to \infty$

$$\hat{g}(-\log\xi) \to 0 \tag{166}$$

providing $g < (4-D)/2a$, (167)

while as $\xi \to 0$

$$\hat{g}(-\log\xi) \to (4-D)/2a \tag{168}$$

At D=3 the parameter which naturally appears in perturbation theory is (see (161) for example)

$$\hat{g}(-\log\xi)/(4\pi)^{3/2} \xrightarrow[\xi\to 0]{} 0.05 . \tag{169}$$

Looking at the perturbation series we see that as $p^2 \to \infty$, the correction terms to $\Gamma^{(2)}=p^2$ or $\Gamma^{(4)}=-\lambda_0$ vanish rapidly. This makes good physical sense since as $p^2 \to \infty$, in $\langle\phi(\vec{x})\rangle$, $|\vec{x}| \to 0$. At very short distances from the source, the influence of the fluctuating medium should be completely negligible; and this analysis affirms that. More precisely, we learn that the physical realm of g is between

$$0 \le \hat{g}(-\log \xi) \le (4-D)/2a \tag{170}$$

and, in the present approximation, that requires

$$\hat{g}(0) = g \le (4-D)/2a$$

The physics of the situation chooses where $\tilde{g}(0)$ must lie. If $\hat{g}(0) > (4-D)/2a$, then as $\xi \to \infty$, $\hat{g}(-\log \xi)$ would blow up.

What is most remarkable about this result is not its ability to reproduce the $p^2 \to \infty$ limit, but what it says about the $p^2 \to 0$ limit; that is, the behavior of $\langle\phi(\vec{x})\rangle$ for very large distances from source to receiver, where the influence of the fluctuating medium is essential as it has the opportunity to "act" many, many times. Our result says that even in this limit, the effective dimensionless expansion parameter at D=3 is small, and perturbation theory for $\Gamma^{(2)}$ (or any $\Gamma^{(N)}$ for that matter) is accurate.

Rather than labor on this point, I want to make some general remarks about the structure of $\tilde{g}(\tau)$ given A(g). The important feature of A(g) in finding the behavior of $\hat{g}(-\log \xi)$

for $\xi \to 0$ or $\xi \to \infty$ is the location of zeroes. Suppose A(g) has a zero at $g=g_1$ where $A'(g_1) > 0$. Then near $g=g_1$ we have

$$\frac{d\hat{g}(\tau)}{d\tau} = -A'(g_1)\left(\hat{g}(\tau)-g_1\right) \tag{172}$$

and

$$\hat{g}(-\log \xi) = g_1 + \xi^{+A'(g_1)}\left(g-g_1\right) \tag{173}$$

As $\xi \to 0$, $\hat{g}(-\log \xi) \to g_1$. Such a behavior has a name: g_1 is called a fixed point of the non-linear differential equation for $\hat{g}(\tau)$. In this case we have a <u>long</u> <u>distance</u> or <u>infrared</u> ($k \to 0$) fixed point. if $A'(g_1) < 0$, we have a <u>short</u> <u>distance</u> fixed point, $\hat{g}(-\log \xi) \to g_1$ as $\xi \to \infty$.

For our A(g) we have two zeroes: g=0 where the slope $A'(0) = -(4-D)/2$ —so this is a short distance fixed point, governing the $p^2 \to \infty$ limit of the theory. We also have a zero at $g_1 = (4-D)/2a$ where $A'(g_1) = (4-D)/2$; this is a long distance fixed point that determines the behavior of correlation functions as $p^2 \to 0$.

We will end this lecture on the renormalization group with two remarks. First I have said that using the differential equations of the renormalization group one can improve on perturbation theory. I want to exhibit this explicitly now. If we calculate Z(g) in perturbation theory, we find

$$Z(g) = 1 + z_0 g^2 + \cdots \tag{174}$$

From Z(g) we can determine C(g) to be

$$C(g) = c\, g^2 + O(g^3) . \tag{175}$$

Z(g), however, satisfies a renormalization group equation

$$\frac{\partial}{\partial g} \log Z(g) = C(g) / A(g) , \tag{176}$$

which follows directly from (138) and (139). With the boundary condition Z(0) = 1, we may integrate this

$$Z(g) = \exp \int_0^g dx \quad C(x) / A(x) , \tag{177}$$

and in this formula use our knowledge of $A(x) = -(4-D)x/2 + ax^2$ and $C(x) = cx^2$ to write

$$Z(g) = \left(1 - g/g_1\right)^{cg_1/a} \; e^{\frac{c}{a} g} \tag{178}$$

valid for $g < g_1$. When g_1 is small, we can even have high confidence in the numerical accuracy of the approximations to C(x) and A(x). In any case (178) certainly is better than a power series in g. In particular in the crucial region around $g \lesssim g_1$ where perturbation theory is really unbelievable, this formula gives a representation of Z(g). In the next lectures when we come to the details of turbulent flow, we will probe this technique further to construct functions such as $\Gamma^{(2)}(p^2)$. Z(g) is the simplest example around, so we'll be content with that right now.

Lastly, I rpomised to comment on the restoration of k_0 and, if one desires, the scales in $\Gamma(\vec{x})$ —the fluctuation correlation function. The method of attack is really identical, except we now have two, g and r, or more dimensionless parameters on which dimensionless functions such as $A(g,r)$ and $C(g,r)$ depend. This means we will have two effective parameters $\tilde{g}(\tau)$ and $\tilde{r}(\tau)$, and we must search a two dimensional parameter space for limits as $\xi \to 0$ or $\xi \to \infty$. The problems involved are straightforward and are explorable by the routes we have indicated. In general, the physics will change as the number of parameters is increased, and one must carefully examine each situation. In particular, the physical boundary conditions such as above where as $\xi \to \infty$ (short distances), $\hat{g}(-\log \xi) \to 0$, must be attended to as one will often find disjoint limit sets of the parameter space; only one will be relevant to the physical situation at hand.

This completes the material for the second lecture. The essential point which I wanted to convey is a method for examining the importance of the non-linear terms in a field theory such as the ones generated by scalar waves in random media or turbulent flow. By identifying an effective dimensionless expansion parameter we have in essence made a mapping from the parameters which appear in the generating functional, such as λ_0 or k_0, to a set of quantitites in which a perturbation series will often converge much more rapidly. The mappings like $g(\lambda_0, \mu)$ will be explored more throughly in the coming lectures.

In the next two lectures I will turn my attention to a detailed exposition of the perturbation theory and renormalization group analysis of stationary and non-stationary turbulent flow of an isotropic, homogeneous incompressible fluid. The development will rest to a large extent on the ideas discussed in this lecture and on the generating functional in Equations (102) and (103).

REFERENCES FOR LECTURE 2

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LECTURES 3 AND 4 The Renormalization Group Applied to Homogeneous, Isotropic Turbulence

In this second set of two lectures I want to explore in some detail the use of the ideas and methods introduced above applied to the study of homogeneous, isotropic turbulent flow of an incompressible fluid. This is a somewhat idealized problem, but contains a large amount of symmetry: spatial translation invariance and rotational invariance, thus making the complicated calculations a bit easier. Very good reviews of the kinematics and the experimental situation in homogeneous, isotropic turbulence are in Batchelor (1953) and Monin and Yaglom (1975). Although such flows are clearly not exactly correct in naturally occurring phenomena—the oceans and atomospheres are stratified, are rotating, and have boundaries—flows in wind tunnels where the turbulence is produced by a grid do seem to approximate isotropy.

Our plan here is as follows: (1) we will discuss the generation of isotropic turbulence by an external randum mixing force and try to make a few useful statements about such forces. One of our tools will be to consider non-stationary flows a long time after the mixing has occurred. Batchelor and Proudman (1956), Saffman (1967) and others (see Monin and Yaglom (1975), Sections 15.3-15.5) have shown that at very long times into the decay period one is exploring the small wave number behavior of the velocity or other correlation functions. We will subsequently need this behavior, so we record the appropriate results at the outset. (2) Armed with a discussion of an essential physical point, we turn to stationary, homogeneous isotropic turbulence and study the perturbation theory in the non-linearity, $\vec{v}\cdot\nabla v_j$, of the Navier-Stokes equation. We will see that the perturbation theory is fine for very large wave numbers, but fails badly when either the wave number is small or when the Reynolds number is large and the wave number is neither small nor large. (3) We will then carry out a renormalization group analysis of this perturbation theory. During this we will introduce a renormalized Reynolds number which is a certain mapping of the original Reynolds number. This renormalized object will turn out to remain finite even when the original quanitity becomes enormous. So it will clearly be a better expansion parameter then the "bare" Reynolds number. At the same time we will discover that the problems we were having at small wave number are "cured". After all this we will utilize the full power of the renormalization group method to give a procedure for constructing the correlation functions of the stochastic velocity field. In particular we will consider the two point correlation $\langle v_j(\vec{x},t)\, v_\ell(\vec{y},\tau)\rangle$ and explore the information it contains on the energy spectral function E(k).

A bibliographical note here: the perturbation series as we shall write it was given a number of years ago by

Wyld (1961) and studied in some detail by Martin, Rose, and Siggia (1973). It is contained in various forms in Kraichnan's work (reviewed by Leslie(1973)). The small wave number regime of the theory was first studied with the renormalization group by Forster, Nelson, and Stephen (1977) and independently by Abarbanel (1978a).

(4) The last topic we will take up will be a return to non-stationary flow. This is interesting in itself, and, in addition, allows us to justify a number of items in our first pass at non-stationary turbulence.

We begin then with the Navier-Stokes equation

$$\frac{\partial}{\partial t} v_j + \vec{v}\cdot\nabla v_j = -\nabla_j p/\rho + \nu_0 \nabla^2 v_j + F_j \tag{179}$$

for an incompressible flow $\nabla_j v_j = 0$. We can eliminate p as usual, and $\nabla_j v_j = 0$ let's us consistently choose $\nabla_j F_j = 0$. The stochasticity in the flow comes from random boundary conditions $v_j(x,T)$ at some initial time T, or, as emphasized by Saffman (1967), equivalently by the operation of a pulse of random forces F_j. Suppose these forces are gaussian with zero mean (required by isotropy anyway) and operate from some initial time $-T_i$ to some final time T_f. Then the correlation function for the forces can be written as

$$\langle F_j(\vec{x},t) F_\ell(\vec{y},\tau)\rangle = \frac{\gamma_0^2}{4} \Delta_{j\ell}(\nabla)\hat{\Gamma}(k_0^2(\vec{y}-\vec{x})^2)\,\delta(t-\tau)\,\theta(t+T_i)\,\theta(T_f-t) \tag{180}$$

In this γ_0 measures the strength of the forces, $\Delta_{j\ell}(\nabla)$ is the operator

$$\Delta_{j\ell}(\nabla) = \delta_{j\ell} - \nabla_j \nabla_\ell / \nabla^2 \tag{181}$$

introduced before. $\tilde{\Gamma}(k_0{}^2x^2)$ contains the information on the spatial dependence of the force correlation, where $k_0{}^{-1}$ is a distance scale connected with the range of the forces. We have taken the correlation to have zero range in time; that is, the forces must act at the same instant or they are uncorrelated. One can modify this by replacing $\delta(t-\tau)$ by another distribution.

The factor $\tilde{\Gamma}(k_0{}^2x^2)$ is chosen to have dimensions (wave number)^D so its fourier transform

$$\Gamma_m(k^2/k_0^2) = \int d^Dx \; e^{-i\vec{k}\cdot\vec{x}} \, \tilde{\Gamma}(k_0^2 x^2) \tag{182}$$

is dimensionless. If the forces have a finite range, then $\tilde{\Gamma}(k_0{}^2x^2)$ will vanish rapidly for k_0x large, and $_m$ will become small for $k > k_0$. This is a good feature from a physical point of view. Energy is then being pumped into the flow at wave numbers k_0 and will be transported by advection ($\vec{v}\cdot\nabla v_j$) to larger wave numbers where it will be dissipated by $\nu_0 \nabla^2 v_j$.

Suppose the forces are strictly zero outside a range $k_0{}^{-1}/2$, so that we might approximate $\tilde{\Gamma}(k_0{}^2x^2)$ by (D=3)

$$\tilde{\Gamma}(k_0^2 x^2) = k_0^3 \, \theta(k_0^{-2} - x^2) \tag{183}$$

This is a bit extreme but will help us see what to expect for finite range forces. From this we find

$$\Gamma_M(k^2/k_0^2) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} k_0^3\, \theta(k_0^{-2} - x^2)$$
$$= \frac{4\pi k_0^3}{k^2}\left(\sin\frac{k}{k_0} - \frac{k}{k_0}\cos\frac{k}{k_0}\right) \tag{184}$$

This oscillates for all k and for large k falls more or less as k^{-2}. At k=0 it is finite, $\Gamma_M(0) = 4\pi/3$. This example illustrates that even for strictly cutoff finite range forces, the k=0 behavior of Γ_M does not reflect that in an easy and obvious fashion. Since k and x are conjugate variables, the $k\to 0$, $x\to\infty$ connection is as usual, but one must be cautious. To take a smoother example, consider

$$\hat{\Gamma}(k_0^2 x^2) = k_0^3\, e^{-k_0|\vec{x}|} \tag{185}$$

for which

$$\Gamma_M(k^2/k_0^2) = 8\pi\left(\frac{k^2}{k_0^2} + 1\right)^{-2} \tag{186}$$

Again Γ_M falls rapidly as $k\to\infty$, and $\Gamma_M(0) \neq 0$. I emphasize these simple examples since the behavior of $\Gamma_M(0)$ will soon be an issue, and in any case it is useful to recognize that short range forces and correlation functions in real space do not require the $k\to 0$ limit of the correlation function in wave number space to vanish. So $\Gamma_M(0) \neq 0$ is not a statement of enormous distances playing a role in the forces generating the turbulent flow.

Now we want to imagine that after the mixing has ceased at $t = T_f$, the flow develops under its own steam transfering energy via $\vec{v}\cdot\nabla v_j$ up to higher wave numbers where $\nu_0 k^2$ dissipates the kinetic energy into heat. For $t \gg T_f$, it is natural to expect the flow to have significantly decayed so v_j is small and the non-linearity is playing a negligible role. The effective equation of motion in this late state of decay is then

$$\left(\frac{\partial}{\partial t} - \nu_0 \nabla^2\right) v_j(\vec{x},t) = F_j(\vec{x},t) \tag{187}$$

From this and the force correlation function (180) we can evaluate the velocity-velocity correlation function

$$\Phi_{j\ell}^{(2,0)}(t,\tau,\vec{k}) = \int d^D x \, e^{i\vec{k}\cdot\vec{x}} \langle v_j(\vec{x},t)\, v_\ell(0,\tau)\rangle \tag{188}$$

$$= \Delta_{j\ell}(k)\, \Phi^{(2,0)}(t,\tau,k^2), \tag{189}$$

where

$$\Delta_{j\ell}(k) = \delta_{j\ell} - k_j k_\ell / k^2 \quad , \tag{190}$$

and the real information lies in the scalar function $\Phi^{(2,0)}(t,\tau,k^2)$. The relation of this scalar function to the energy spectrum function $E(k,t)$ which gives the energy in the flow in the wave number interval k to $k+dk$ (Batchelor (1953)) is

$$E(k,t) = \frac{D-1}{\Gamma(D/2)} \frac{k^{D-1}}{(4\pi)^{D/2}} \Phi^{(2,0)}(t,t,k^2) \qquad (191)$$

From the linearized Navier-Stokes equation we have

$$\Phi^{(2,0)}(t,\tau,k^2) = \int \frac{d\omega\, d\omega'}{(2\pi)^2} e^{-i\omega t} e^{i\omega'\tau} \frac{\gamma_o^2}{4} \Gamma_m(k^2/k_0^2) \times$$

$$[(-i\omega+\nu_0 k^2)(i\omega'+\nu_0 k^2)]^{-1} \int_{-T_i}^{T_f} d\rho\, e^{i(\omega-\omega')\rho} \quad . \qquad (192)$$

Let's now look at $t > T_f$, so times into the decay period, then

$$\Phi^{(2,0)}(t,t,k^2) = \frac{\gamma_o^2}{4} \Gamma_m(k^2/k_0^2) \frac{e^{-2\nu_0 k^2 t}}{2\nu_0 k^2} \left(e^{2\nu_0 k^2 T_f} - e^{-2\nu_0 k^2 T_i} \right). \qquad (193)$$

For large times, where we expect the linear approximation to be accurate, the factor $e^{-2\nu_0 k^2 t}$ kills $\Phi^{(2,0)}$ unless $k^2 \approx 0$. (More precisely $k^2 t \sim O(1)$). Expanding the coefficient of this exponential factor gives

$$\Phi^{(2,0)}(t,t,k^2) \underset{t \gg T_f}{\approx} \frac{\gamma_o^2}{4} (T_f + T_i) \Gamma_M(0)\, e^{-2\nu_0 k^2 t} + O(k^2) . \qquad (194)$$

The role of $\Gamma_M(0)$ is now explicit.

Batchelor (1953) argues that the velocity correlation function $\Phi_{ij}^{(2,0)}$ should be analytic near k=0. Since $\Phi_{ij}^{(2,0)}$ has a factor $\Delta_{ji}(k)$, this requires $\Phi^{(2,0)}$, the scalar correlation, to behave as k^2 to the first or higher power as $k^2 \to 0$. This means

$$\Gamma_m(k^2) = k^2 h_m(k^2), \qquad (195)$$

where $h_M(0)$ is likely non-zero. Tracing this back to our discussion of $\Gamma_M(0)$, we see that this requires a peculiar oscillatory behavior of the forces in $\vec{x}$ space.

On the other hand Saffman (1967) suggests that it is not the velocity correlation function which should be analytic at k=0, but the correlation function of <u>vorticity</u>, that is, curl $\vec{v}$. This introduces two more powers of k into the discussion and leads to the result that if $\Gamma_M(0) \neq 0$, the vorticity-vorticity correlation function is analytic, while $\Phi_{jl}^{(2,0)}$ is not. Saffman then shows that under his assumption, the quantity $\gamma_0^2 \Gamma_M(0)$ is an invariant of the motion while if $\Gamma_M(q^2) = q^2 h_M(q^2)$, $\gamma_0^2 h_M(0)$ is not.

Furthermore, when $\Gamma_M(0) \neq 0$ we see that the energy spectrum for $t >> T_f$ behaves as

$$E(k,t) \underset{t>>T_f}{\approx} k^{D-1} \qquad (196)$$

for small k. At these times, long after the mixing force has been turned off, the degrees of freedom of the motion should have come to some kind of equilibrium with each other and the equipartition theorem would then suggest the behavior (196).

From a variety of points of view, then, albeit all are heuristic, $\Gamma_M(0) \neq 0$ is suggestive as a general behavior of the wave number space correlation function of the mixing forces. We'll choose $\Gamma_M(0) = 1$ as our standard convention, throwing all other factors into $\gamma_0^2/4$.

As we turn to a discussion of stationary turbulence, it is useful to investigate the behavior of our $\Phi^{(2,0)}(t,t,k^2)$ for $-T_i<t<T_f$ as $T_i,T_f\to\infty$. This means that the mixing force is on always, but that is precisely what we need for stationarity of the flow, for turning off the mixing allows viscous decay to occur. When $-T_i<t<T_f$, we find from (192)

$$\Phi^{(2,0)}(t,t,k^2) = \frac{1-e^{-2\nu_0 k^2(t+T_i)}}{2\nu_0 k^2}\,\Gamma_m(k^2/k_0^2)\,\frac{\chi_0^2}{4}\,. \tag{197}$$

So as $T_i,T_f\to\infty$,

$$\Phi^{(2,0)}(t,t,k^2) \sim \frac{\chi_0^2}{8\nu_0 k^2}\,\Gamma_m(k^2/k_0^2), \tag{198}$$

which is independent of t as it must be for a stationary flow. In this situation the energy spectrum is

$$E(k) \sim k^{D-3}\,\Gamma_m(k^2/k_0^2) \quad , \tag{199}$$

but one must caution that this result holds only when the linearized Navier-Stokes equation is a good approximation. Nevertheless, we see an important factor of k^{-2} entering here; a factor which is canceled in the non-stationary flow at $k^2\to 0$. Since the fluid is being mixed constantly and predominantly at $k\lesssim k_0$, one would not expect equipartition to necessarily apply. Nonetheless, if $\Gamma_M(k^2)\sim k^2$, it would appear to do that. But this is a bit artificial, since if $\Gamma_M(k^2)\sim k^4$, the mixing again is strongly absent at small wave numbers, but $E(k)\sim k^{D+1}$. Forster, Nelson, and Stephen (1977) argue in favor of $\Gamma_M(k^2)\sim k^2$ using the $E(k) \sim k^{D-3}\Gamma_M(k^2)$ connection for stationary turbulence. They connect this to Saffman's results, but as we have seen there is an extra k^2 factor in the non-stationary case, so one must exercise caution.

We now turn to stationary, homogeneous, isotropic turbulent flow and develop the perturbation theory for it when the flow is mixed by a gaussian random force. Our task, simply stated, is to evaluate the functional integral (102) for the generating functional for all velocity and anti-velocity correlation functions $\Phi^{(n,m)}$:

$$\langle v_{j_1}(\vec{x}_1,t_1)\cdots v_{j_n}(\vec{x}_n,t_n)\,\bar{v}_{\ell_1}(\vec{y}_1,\tau_1)\cdots \bar{v}_{\ell_m}(\vec{y}_m,\tau_m)\rangle . \tag{200}$$

So much for the statement of the problem, now for the work.

We begin by looking at the effective lagrangian

$$\mathcal{L}(v_j,\bar{v}_j) = \frac{1}{2}\bar{v}_j \overleftrightarrow{\partial}_t v_j + \nu_0 \nabla_n \bar{v}_j \nabla_n v_j - \frac{\gamma_0^2}{8}\bar{v}_j \tilde{\Gamma} \bar{v}_j - \frac{1}{2}\Big(\big(\Delta_{jn}\nabla_\ell + \Delta_{j\ell}\nabla_n\big)\bar{v}_j\Big) v_n v_\ell . \tag{201}$$

If we make the scaling

$$\chi_j = 2 v_j/\gamma_0 \tag{202}$$

and

$$\bar{\chi}_j = \gamma_0 \bar{v}_j/2 \tag{203}$$

we find

$$\mathcal{L}(\chi_j,\bar{\chi}_j) = \frac{1}{2}\bar{\chi}_j \overleftrightarrow{\partial}_t \chi_j + \nu_0 \nabla_n \bar{\chi}_j \nabla_n \chi_j - \frac{1}{2}\bar{\chi}_j \tilde{\Gamma}\bar{\chi}_j - \frac{\gamma_0}{4}\Big(\big(\Delta_{jn}\nabla_\ell + \Delta_{j\ell}\nabla_n\big)\bar{\chi}_j\Big)\chi_n \chi_\ell , \tag{204}$$

which puts the γ_0 in front of the non-linear term and shows how the scale of the random force sets the magnitude of the non-linear effects.

The action

$$\int d^Dx\, dt\, \mathcal{L}(\chi_j, \bar{\chi}_j) \tag{205}$$

is dimensionless. We can exhibit the dimensions of the ingredients in $\mathcal{L}$ by writing dimensions in wave number, k, and frequency, ω , units. Since

$$[x] = k^{-1} \tag{206}$$

and

$$[t] = \omega^{-1} \tag{207}$$

we deduce from (205)

$$[\chi_j(\vec{x},t)] = k^{D/2}\,\omega^{-1/2}, \tag{208}$$

$$[\bar{\chi}_j(\vec{x},t)] = k^{D/2}\,\omega^{1/2}, \tag{209}$$

$$[\nu_0] = \omega k^{-2}, \tag{210}$$

and

$$[\gamma_0] = \omega^{3/2}\,k^{-\left(\frac{D+2}{2}\right)}. \tag{211}$$

Of course, $[k_0] = k$. γ_0 is going to act as our expansion parameter, but as in the previous example of scalar wave propagation, it is not dimensionless. The combination

$$R_0 = \frac{\gamma_0}{\nu_0^{3/2}}\,k_0^{\frac{D-4}{2}} \tag{212}$$

is dimensionless and is the Reynolds number based on the external length scale k_0^{-1}. To see this we note that using k_0 as the wave number scale, the dimensions of v_j are

$$[v_j(\vec{x},t)] = [\gamma_0]\, k_0^{D/2-1} \Big/ [\nu_0]^{3/2} \qquad (213)$$

using $[\omega] = [\nu_0] k_0^2$. So Reynolds number would be

$$R_0 = \frac{v}{\nu_0 k_0} = \frac{\gamma_0 \, k_0^{\frac{D-4}{2}}}{\nu_0^{3/2}} \qquad (214)$$

as noted.

However, we want to introduce a wave number scale into the problem, call it k_N, which we can vary to explore the response to changes in k^2 when γ_0, ν_0, and k_0 are fixed. So we have two dimensionless quantities

$$g_0 = \frac{\gamma_0}{\nu_0^{3/2}} \left(k_N^2\right)^{\frac{D-4}{4}} , \qquad (215)$$

which will be the "bare" dimensionless expansion parameter, and

$$\sigma = k_N^2 / k^2 \qquad (216)$$

which measures the scale of wave number relative to the external length in the theory.

With a dimensionless parameter in mind we can begin looking into perturbation theory in γ_0. If we set $\gamma_0=0$, then $\mathcal{L}(\chi_j, \bar{\chi}_j)$ is at most quandratic in fields, so we can do the χ_j and $\bar{\chi}_j$ integrations to find the generating function $Z[\eta_j, \bar{\eta}_j]$ for χ_j, $\bar{\chi}_j$ correlation functions.

Only two are non-zero in lowest order of γ_0. They are

$$G_{0j\ell}^{(1,1)}(\vec{k},\omega) = \int d^Dx\,dt\; e^{i(\vec{k}\cdot\vec{x}-\omega t)} \langle \chi_j(\vec{x},t)\bar{\chi}_\ell(0,0)\rangle , \tag{217}$$

$$= \Delta_{j\ell}(k)\, \mathcal{G}_0^{(1,1)}(k^2,\omega), \tag{218}$$

$$= \Delta_{j\ell}(k)\left[-i\omega + \nu_0 k^2\right]^{-1} , \tag{219}$$

and

$$G_{0j\ell}^{(2,0)}(\vec{k},\omega) = \int d^Dx\,dt\; e^{i(\vec{k}\cdot\vec{x}-\omega t)} \langle \chi_j(\vec{x},t)\chi_\ell(0,0)\rangle , \tag{220}$$

$$= \Delta_{j\ell}(k)\, \mathcal{G}_0^{(2,0)}(k^2,\omega), \tag{221}$$

$$= \Delta_{j\ell}(k)\,\Gamma_m(k^2/k_0^2)\left[(-i\omega+\nu_0 k^2)(i\omega+\nu_0 k^2)\right]^{-1}. \tag{222}$$

The factor $\Delta_{j\ell}(k)$ in each of these comes from the isotropy and homogeneity of the system and from $\nabla_j \chi_j = \nabla_j \bar{\chi}_j = 0$.

To develop a diagramatic expression for the expansion in γ_0 we will associate a solid line with each factor χ_j and a dotted line with each $\bar{\chi}_j$. Since there is a definite direction in time, we associate an arrow going to the left with a positive frequency ; an arrow to the right, a negative frequency. This gives the diagram elements in Fig. 8.

At the next order in γ_0 we find the interaction vertex where a χ_j of $\vec{k}_1, \omega_1$ and a χ_ℓ of $\vec{k}_2, \omega_2$ fuse to a $\bar{\chi}_n$ of $\vec{k} = \vec{k}_1 + \vec{k}_2$, $\omega = \omega_1 + \omega_2$

$$G_{0nj\ell}^{(1,2)}(\vec{k},\omega,\vec{k}_1,\omega_1,\vec{k}_2,\omega_2) =$$

$$G_{0j\alpha}^{(1,1)}(\vec{k}_1,\omega_1)\; G_{0\ell\beta}^{(1,1)}(\vec{k}_2,\omega_2)\; G_{0n\nu}^{(1,1)}(\vec{k},\omega) \times$$

$$\times \frac{-i\gamma_0}{2(2\pi)^{\frac{D+1}{2}}}\left(\delta_{\alpha\nu}k_\beta + \delta_{\beta\nu}k_\alpha\right) \quad . \tag{223}$$

This suggest we define a one line irreducible vertex $\Gamma_{0\nu,\alpha\beta}(\vec{k},\omega,\vec{k}_1,\omega_1,\vec{k}_2,\omega_2)$ by amputating the $G^{(1,1)}$ factors. The lowest order term in Γ is shown in Fig. 9.

Studies of higher order correlation functions and higher orders in perturbation theory in γ_0 show that $G_0^{(1,1)}$, $G_0^{(2,0)}$, and Γ_0 are the ingredients needed to construct any graph desired. The graphical rules that emerge are these:

1. At any given order of γ_0 draw all topologically distinct graphs constructed out of $G_0^{(1,1)}$, $G_0^{(2,0)}$, and Γ(Fig. 9).

2. Integrate $d^Dk\, d\omega$ around each closed loop.

3. At each vertex conserve $\vec{k}$ and ω .

4. With each graph associate a weight of unity except for closed loops containing two factors of $G_0^{(1,1)}$ - these have weight $\frac{1}{2}$.

You will find these rules in Wyld (1961) and more explicitly in Martin, Rose, and Siggia (1973), the appendix of Foster, Nelson, and Stephen (1977) or Abarbanel (1978a). Martin, Rose, and Siggia (1973) also discuss the integral equations

(Dyson equations) which represent the formal sum of perturbation theory.

To order $\gamma_0^{\,2}$ we can evaluate $G_{jl}^{\,(1,1)}(\vec{k},\omega)$ by calculating the graphs shown in Fig. 10. A factor $\Delta_{jl}(k)$ can always be extracted from G_{jl} so

$$G_{jl}^{(1,1)}(\vec{k},\omega) = \Delta_{jl}(k)\,\mathcal{Y}^{(1,1)}(k^2,\omega)\,. \tag{224}$$

$\mathcal{Y}^{(1,1)}$ may be written as

$$\mathcal{Y}^{(1,1)}(k^2,\omega)^{-1} = -i\omega + \nu_0 k^2 - \Sigma(k^2,\omega) \tag{225}$$

where to order $\gamma_0^{\,2}$

$$\Sigma(k^2,\omega) = -\frac{\gamma_0^{\,2}}{4(2\pi)^{\frac{D+1}{2}}}\int \frac{d^Dq\, d\omega'\, \Gamma_m(q^2/k_0^2)}{(-i\omega+\nu_0 q^2)(i\omega'+\nu_0 q^2)\left[-i(\omega-\omega')+\nu_0(\vec{q}-\vec{k})^2\right]}\times$$

$$\times\left[\frac{k^2}{q^2} - \frac{2\vec{k}\cdot(\vec{k}-\vec{q})}{(D-1)(\vec{k}-\vec{q})^2}\right] q_j\,\Delta_{jl}(k)\,q_l\,, \tag{226}$$

which on doing the ω' integral becomes

$$\Sigma(k^2,\omega) = -\frac{\gamma_0^{\,2}}{4(2\pi)^D}\int \frac{d^Dq\, \Gamma_m(q^2/k_0^2)}{2\nu_0 q^2\left[-i\omega+\nu_0(q^2+(\vec{q}-\vec{k})^2\right]}\, q_j\,\Delta_{jl}(k)\,q_l\,\times$$

$$\times\left[\frac{k^2}{q^2} - \frac{2\vec{k}\cdot(\vec{k}-\vec{q})}{(D-1)(\vec{k}-\vec{q})^2}\right]. \tag{227}$$

If $k^2 \to \infty$ with γ_0, ν_0, k_0 fixed in this formula, $\Sigma \to$ constant. This means it becomes negligible relative to the $-i\omega+\nu_0 k^2$ in $\left(\mathcal{Y}^{(1,1)}\right)^{-1}$. In this limit then, the corrections to the lowest order term in $\mathcal{Y}^{(1,1)}$ are not important and the dissipative and kinetic terms dominate over the advective. The effective Reynolds number when $k^2 \to \infty$ is very small.

In the limit $k^2 \to 0$, $\omega \to 0$ we must exercise a bit more care. Since Σ vanishes as k^2 in this limit as does $-i\omega + \nu_0 k^2$, we should extract a factor of k^2 before proceeding. Then we find that the dimensionless form

$$\frac{\Sigma}{\nu_0 k^2} \underset{\substack{k^2 \to 0 \\ \omega \to 0}}{\sim} R_0^2 \int \frac{d^D q}{q^4} \, \Gamma_M(q^2) \quad , \qquad (228)$$

For any γ_0, ν_0, and k_0, when $\Gamma_M(0) \neq 0$, this integral is divergent for small $\vec{q}$ in $D \leq 4$ dimensions. The correction term to $-i\omega + \nu_0 k^2$, then swamps the original term and perturbation theory is completely incorrect.

Of course, if R_0^2 is very large numerically, then for finite k^2, ω the correction to $-i\omega + \nu_0 k^2$ is numerically large and a simple perturbation expansion in it is not wise. Regardless of the numerical size of R_0, since it is a fixed external parameter, when k becomes large enough, $\Sigma/\nu_0 k^2 \to 0$. Similarly, regardless of how small R_0 may be (as long as it's non-zero), $\Sigma/\nu_0 k^2$ blows up for $k^2 \to 0$ in $D<4$.

This is really a crucial stage in our development. We see in the simplest example of perturbation theory, that the quality of the expansion depends on the regime of phase space and the dimensions of space as well as the numerical value of R_0. For R_0 large, we eventually have a reasonable perturbation theory, when k goes deep enough into the dissipation regime. Our efforts will now be focused on dealing with the $k^2 \to 0$ behavior keeping in mind the "boundary condition" that whatever we do must yield the known $k^2 \to \infty$ limit. Once we have control over these two ends of the wave number space, we can attempt to assess the success we have had in the intermediate k range. One can anticipate, but not guarantee, of course, that tying down the $k^2 \to 0$ and $k^2 \to \infty$ behavior of a rather smooth function in one smooth formula, will yield good results in the middle.

Before launching into the renormalization group cure, let us make a remark about where large and small k lie relative to the usual external and internal scales of turbulence. Novikov (1964) has shown that the net energy dissipation due to viscosity, called ε, is

$$\varepsilon = \frac{D-1}{2} \frac{\gamma_0^2}{4} \hat{\Gamma}(0) \tag{229}$$

$$= \frac{D-1}{2} \frac{\gamma_0^2}{4} \int \frac{d^D p}{(2\pi)^D} \Gamma_m (p^2/k_0^2)$$

$$= \gamma_0^2 k_0^D \times \text{constant}. \tag{230}$$

The Kolmogorov turbulent length scale η is then

$$\eta^{-1} = (\varepsilon/\nu_0^3)^{1/4} = k_0 \sqrt{R_0}\,. \tag{231}$$

Small k clearly means $k \ll k_0$ so one is in the production range of wave number. Large k must mean $k\eta \gg 1$ or $k \gg k_0 \sqrt{R_0}$ so one is in the deep dissipation range. The inertial range will lie in $k_0 \ll k \ll \eta^{-1} = k_0 \sqrt{R_0}$ as usual and to have a sizeable inertial range requires R_0 large- again, as usual. So when I speak of very large k, I am not talking about the inertial range. That intermediate range of k must be derived from our procedure.

Now we are ready to begin the renormalization group analysis. As in the scalar wave propagation discussion before we find the interaction of modes via the $\vec{v}\cdot\nabla v_j$ term rescales the ingredients of the effective lagrangian which here are ν_0, γ_0, χ_j and $\bar{\chi}_j$. (k_0 is rescaled, but not in a manner essential to our discussion.) We define, as before, a scaling of the fields and constants so

$$\chi_{Rj}(\vec{x},t) = Z^{-1/2}\,\chi_j(\vec{x},t)\,, \tag{232}$$

$$\bar{\chi}_{Rj}(\vec{x},t) = Z^{-1/2}\,\bar{\chi}_j(\vec{x},t)\,, \tag{233}$$

$$\nu = Z_\nu\,\nu_0\,, \tag{234}$$

and

$$\gamma = Z_\gamma\,\gamma_0\,. \tag{235}$$

The renormalized expansion parameter

$$g = \frac{\gamma}{\nu^{3/2}}\,k_N^{\frac{D-4}{2}} = \frac{Z_\gamma}{Z_\nu^{3/2}}\,g_0 = \mathcal{Z}\,g_0 \tag{236}$$

will be a function of g_0 and σ which we will determine shortly. If we are fortunate, this new expansion parameter will provide an improved perturbation series. Indeed we will show that as g_0 ranges over $0 \leq g_0 < \infty$, g remains between

$0 \leq g \leq$ finite number for $D<4$. Even if λ_0 is very large, so R_0 is very large, g remains bounded.

To determine the scaling factors we need some normalization conditions on the renormalized correlation functions $G_R^{(n,m)} = \langle \chi_{Rj_1} \cdots \chi_{Rj_n} \bar{\chi}_{R\ell_1} \cdots \bar{\chi}_{R\ell_m} \rangle$ at some convenient point in $\vec{k}$, ω space. We will choose the following:

1. On $\mathcal{G}_R^{(1,1)}(k^2,\omega) = (Z\bar{Z})^{-1/2}\mathcal{G}^{(1,1)}(k^2,\omega)$ we require

$$\frac{\partial}{\partial\omega}\mathcal{G}_R^{(1,1)}(k^2,\omega)^{-1}\bigg|_{\substack{k^2=0 \\ \omega=i\nu k_N^2=i\omega_N}} = -i, \tag{237}$$

and

$$\frac{\partial}{\partial k^2}\mathcal{G}_R^{(1,1)}(k^2,\omega)^{-1}\bigg|_{\substack{k^2=0 \\ \omega=i\nu k_N^2=i\omega_N}} = \nu = Z_\nu \nu_0 . \tag{238}$$

This determines $Z\bar{Z}$ and Z_ν as a series in λ_0 once we have calculated $\mathcal{G}^{(1,1)}(k^2,\omega)$ as a series in λ_0. These conditions are suggested by the value of $\mathcal{G}^{(1,1)-1}$ when $\lambda_0=0$: $-i\omega+\nu_0 k^2$ and imply $Z\bar{Z}, Z_\nu = 1 + O(\lambda_0^2)$.

2. If we write $G_{Rij}^{(2,0)}(\vec{k},\omega) = \Delta_{ij}(k)\mathcal{G}_R^{(2,0)}(k^2,\omega)$ then $\mathcal{G}_R^{(2,0)}(k^2,\omega) = Z^{-1}\mathcal{G}^{(2,0)}(k^2,\omega)$. Z is determined by requiring

$$\frac{\partial}{\partial\omega}\mathcal{G}_R^{(2,0)}(k^2,\omega)^{-1}\bigg|_{\substack{k^2=0 \\ \omega=i\nu k_N^2=i\omega_N}} = 2i\nu k_N^2 , \tag{239}$$

once we know $\mathcal{G}^{(2,0)}(k^2,\omega)$ as a series in λ_0.

3. Looking at the lowest order contribution to $\Gamma_{\nu,\alpha\beta}(\vec{k},\omega,\vec{k}_1,\omega_1,\vec{k}_2,\omega_2)$; namely,

$$\Gamma_{0\nu,\alpha\beta}(\vec{k},\omega,k_1,\omega_1,k_2,\omega_2) = \frac{-i\gamma_0}{2(2\pi)^{\frac{D+1}{2}}}\left[\delta_{\nu\alpha}k_\beta + \delta_{\nu\beta}k_\alpha\right], \qquad (240)$$

we determine γ or Z_γ by observing $\Gamma_{R\nu;\alpha\beta} = Z\bar{Z}^{1/2}\,\Gamma_{\nu;\alpha\beta}$ where $\Gamma_{\nu;\alpha\beta}$ is the fusion vertex computed to all orders in γ_0 and requiring

$$\left.\frac{k_\nu\Gamma_{R\nu;\alpha\beta}(\vec{k},\omega,\vec{k}_1,\omega_1,\vec{k}_2,\omega_2)\delta_{\alpha\beta}}{k^2}\right|_{\substack{\vec{k}_1=\vec{k}_2=\vec{k}=0\\ \omega=2\omega_1=2\omega_2=i\nu k_N^2=i\omega_N}} = -i\gamma\Big/(2\pi)^{\frac{D+1}{2}}. \qquad (241)$$

This procedure enables us to develop a renormalization group equation for the renormalized n-velocity m-anti-velocity correlation functions $\Phi^{(n,m)}$. The procedure is almost identical to the steps carried out above. Life is made slightly simpler by noting that the way we have chosen the conditions for defining Z and $\bar{Z}$, namely normalizing at $k^2=0$, Z and $\bar{Z}$ are each 1(Abarbanel (1978a, 1978b)). So $\Phi_R^{(n,m)} = (\gamma/2)^{n-m}\,G_R^{(n,m)}$ satisfies the <u>renormalization group equation</u>

$$\Big[k_N^2\frac{\partial}{\partial k_N^2} + \frac{A(g,\sigma)}{1-B(g,\sigma)}\frac{\partial}{\partial g} + \frac{B(g,\sigma)}{1-B(g,\sigma)}\nu\frac{\partial}{\partial\nu} + \sigma\frac{\partial}{\partial\sigma}$$
$$+ C_{n,m}(g,\sigma)\Big]\,\Phi_R^{(n,m)}(\vec{k}_i,\omega_i,g,\nu,\sigma,k_N^2) = 0 \qquad (242)$$

where

$$A(g,\sigma) = \omega_N\frac{\partial}{\partial\omega_N}g\Big|_{\gamma_0,\nu_0,k_0\ \text{fixed}}, \qquad (243)$$

$$B(g,\sigma) = \frac{\omega_N}{\nu} \frac{\partial}{\partial \omega_N} \nu \Big|_{\gamma_0, \nu_0, k_0 \text{ fixed}} , \qquad (244)$$

and

$$C_{n,m}(g,\sigma) = (m-n)\left[1 - \frac{D}{4} + \frac{D+2}{4} B(g,\sigma) + \frac{A(g,\sigma)}{g}\right] . \qquad (245)$$

These express the independence of $G^{(n,m)}(\vec{k}_i, \omega_i, \gamma_0, \nu_0, k_0)$ from the normalization point $\omega_N = \nu k_N^2$ and the chain rule. In the present problem ω_N or k_N^2 plays the same role as μ^2 in the scalar propagation problem above.

Once again we can solve the renormalization group equation by the method of characteristics. To this end introduce the <u>effective expansion parameter</u> $\tilde{g}(\tau)$, the <u>effective viscosity</u> $\tilde{\nu}(\tau)$, and the <u>effective scale ratio</u> $\tilde{\sigma}(\tau)$ which obey

$$\frac{d\tilde{g}(\tau)}{d\tau} = -A(\tilde{g}(\tau), \tilde{\sigma}(\tau)) \Big/ \left(1 - B(\tilde{g}(\tau), \tilde{\sigma}(\tau))\right) , \qquad (246)$$

$$\frac{d\tilde{\nu}(\tau)}{\tilde{\nu}(\tau)\, d\tau} = -1 \Big/ \left(1 - B(\tilde{g}(\tau), \tilde{\sigma}(\tau))\right) , \qquad (247)$$

and

$$\frac{d\tilde{\sigma}(\tau)}{d\tau} = -\tilde{\sigma}(\tau) , \qquad (248)$$

with the boundary conditions $\tilde{g}(0) = g$, $\tilde{\nu}(0) = \nu$, $\tilde{\sigma}(0) = \sigma$.

After the conventional dimensional analysis we find

$$\Phi_R^{(n,m)}\left(\sqrt{\xi}\,\vec{k}_i, \omega_i, g, \nu, \sigma, k_N^2\right) =$$

$$\Phi_R^{(n,m)}\left(\vec{k}_i, \omega_i, \tilde{g}(-\log\xi), \hat{\nu}(-\log\xi), \xi\sigma, k_N^2\right) \times$$

$$\times \exp \int_{-\log\xi}^{0} d\tau\, \gamma_{n,m}\left(\tilde{g}(\tau), \sigma e^{-\tau}\right), \tag{249}$$

noting that $\tilde{\sigma}(\tau) = \sigma e^{-\tau}$ and where

$$\gamma_{n,m}(g,\sigma) = \frac{(m-n)(2-D)+2(1-n)D}{4} + \frac{(m-n)}{1-B(g,\sigma)}\left[1 + \frac{B(g,\sigma)}{2} + \frac{A(g,\sigma)}{g}\right]. \tag{250}$$

Once more we have a result which tells us the behavior of our various correlation functions in different regions of $\vec{k}$ space once we know how $\tilde{g}(-\log\xi)$ and $\hat{\nu}(-\log\xi)$ behave. We certainly expect that as $\xi\to\infty$, $\tilde{g}(-\log\xi)\to 0$ as that reflects the excellence of perturbation theory in the deep dissipation region, $k\eta \gg 1$. The $\xi\to 0$ limit, where perturbation theory fell apart, is a central issue.

To determine $A(g,\sigma)$ and $B(g,\sigma)$ we need $\Sigma(k^2,\omega)$ and to lowest order in γ_0 the graphs in Fig. 11. With our normalization condition $Z_\gamma = 1$ to $O(\gamma_0^{\,2})$ as the contribution of the $O(\gamma_0^{\,3})$ graphs in Fig. 11 cancels at our normalization point, Eq. (241). For $B(g,\sigma)$ we find

$$B(g,\sigma) = -\frac{g^2}{D+2}\,\frac{\sigma^{3-D/2}}{32D}\int\frac{d^Dq}{(2\pi)^D}\,\frac{\Gamma_m(q^2)}{q^2}\,\frac{\left[q^2(D^2-D+2)+\sigma/2\,(D^2-D-2)\right]}{(q^2+\sigma/2)^3}, \tag{251}$$

$$= -\frac{g^2}{D+2}\,\frac{1}{32D}\int\frac{d^Dp}{(2\pi)^D}\,\frac{\Gamma_m(\sigma p^2)}{p^2(p^2+1/2)^3}\left[p^2(D^2-D+2)+\frac{1}{2}(D^2-D-2)\right], \quad (252)$$

$$= -\left(g^2/_{D+2}\right)F(\sigma). \quad (253)$$

For $A(g,\sigma)$ we have

$$A(g,\sigma) = -\frac{(4-D)}{4}g + \frac{F(\sigma)}{4}g^3 \quad . \quad (254)$$

Before studying $\tilde{g}(\tau)$ let us find the relation between g and g_0. To do this we need $g/g_0 = \mathcal{Z} = Z_\nu^{-3/2}$, since $Z_\mu =$ From the definitions of A and B we find

$$\frac{\partial}{\partial g}\log Z_\nu = B(g,\sigma)/A(g,\sigma). \quad (255)$$

With the boundary condition $Z_\nu(0,\sigma) = 1$, and our values for A and B, this leads to

$$Z_\nu(g,\sigma) = \left(1-g^2/g_1^2\right)^{-2/D+2}, \quad (256)$$

and

$$g_0^2 = g^2\Big/\left(1-g^2/g_1^2\right)^{6/D+2} \quad (257)$$

in which

$$g_1^2 = (4-D)/F(\sigma) \quad (258)$$

is the zero of $A(g,\sigma)$ with positive slope. For illustration consider this at D=4. Then

$$g^2 = g_0^2\Big/\left(1+g_0^2/g_1^2\right) \quad . \quad (259)$$

In this we see that when g_0, the unrenormalized expansion parameter, is small, so is g. Also we note that as g_0 increases, which means $\gamma_0/\nu_0{}^{3/2}$ and R_0 increase, g grows from zero but only up to g_1. The same behavior is clear in (257) for D-3. This means that our mapping of g_0 into g has taken us from an expansion parameter, g_0, which can become very large when the external parameters $\gamma_0/\nu_0{}^{3/2}$ or the external Reynolds number becomes large, over to another parameter, g, whose range is finite. This does not at all prove the convergence of a series in g. It does, however, open the issue and certainly provides a much tamer series than a direct expansion in g_0 or γ_0.

From (254) we can now study the behavior of $\tilde{g}(\tau)$. First to touch base with our desire to have a dissipation region we look at $\xi = e^{-\tau} \to \infty$, or $\tau \to -\infty$. $\hat{\sigma}(\tau) = \sigma e^{-\tau} \to \infty$ in this limit, and we need the behavior of $F(\tilde{\sigma}(\tau))$ for large argument. From (251) we see

$$F(\sigma) \underset{\sigma\to\infty}{\sim} \frac{\sigma^{1-D/2}}{8D} \int \frac{d^Dq}{(2\pi)^D} \Gamma_M(q^2)/q^2 \quad , \qquad (260)$$

so for D>2, $F(\sigma)$ goes rapidly to zero as long as the coefficient of $\sigma^{1-D/2}$ in (260) exists. That integral also exists for D>2 when $\Gamma_M(q^2)$ goes to zero rapidly for large q^2 and $\Gamma_M(0) = 1$. If $\Gamma_M(q^2) \sim (q^2)^{-\lambda}$ for large q^2, then for $2<D<2(1+\lambda)$, $F(\sigma)\to 0$, $\sigma\to\infty$. In this circumstance, the equation for $\hat{g}(\tau)$ for $\tau \to -\infty$ becomes

$$\frac{d\hat{g}(\tau)}{d\tau} = + \frac{(4-D)}{4}\hat{g}(\tau) \qquad (261)$$

leading to

$$\hat{g}(-\log \xi) \underset{\xi \to \infty}{\sim} \xi^{-(4-D)/4} \tag{262}$$

At D=3 this corresponds precisely to the $O(1/k^2)$ corrections we observed in perturbation theory. So we do find a $\tilde{g}$ which becomes small in the deep dissipation region.

For $\xi \to 0$ we must examine $\hat{\sigma}(\tau) \to 0$ in which limit $F(\sigma)$ becomes

$$F(0) = \frac{1}{32D} \int \frac{d^D p}{(2\pi)^D} \frac{\left[p^2 (D^2 - D + 2) + \frac{1}{2}(D^2 - D - 2) \right]}{p^2 (p^2 + 1/2)^3} \tag{263}$$

which is finite for $2<D<6$. The equation for $\hat{g}(\tau)$ in the region $\tau = -\log \xi \to \infty$ looks like

$$\frac{d\hat{g}(\tau)}{d\tau} = \left(\frac{4-D}{4} \hat{g}(\tau) - \frac{F(0)}{4} \hat{g}(\tau)^3 \right) \bigg/ \left(1 + \frac{F(0)\hat{g}(\tau)^2}{D+2} \right)' \tag{264}$$

As τ increases, $\hat{g}(\tau)$ increases until it hits $g_1^2 = (4-D)/F(0)$, which is the infrared or long distance fixed point of the theory.

We can note a connection between the $g(g_0)$ relation discussed before and the $\hat{g}(\tau)$ behavior here. g_0 is a mixture of γ_0, ν_0 and our normalization wave number k_N

$$g_0^2 = \left(\gamma_0^2 / \nu_0^3 \right) \left(k_N^2 \right)^{\frac{D-4}{2}}$$

As we vary k_N we are probing various parts of wave number space. As $k_N \to 0$, $g_0 \to \infty$ and the unrenormalized expansion parameter is surely not at all useful. Looking again at

(257) we notice that considering k_N as a variable, that $g(k_N)$ is going precisely to $(4-D)/F(0)$ as $k_N \to 0$. Similarly, as $k_N \to \infty$ and we explore the deep dissipation region, $g_0 \to 0$ as does $g(k_N)$. It should be clear by now that discussing the $g(k_N)$ vs. $g_0(k_N)$ behavior as k_N varies is tantamount to studying $\tilde{g}(\tau)$.

Along the way we have solved for Z_ν so we can determine how the effective viscosity behaves as a function of wave number. We have

$$\nu(k_N) = Z_\nu\left(g(k_N)\right) \nu_0 \tag{265}$$

$$= \left(1 - \frac{g^2(k_N)\, F(\sigma(k_N))}{4-D}\right)^{-2/D+2} \nu_0 \tag{266}$$

as $k_N \to 0$, $g^2(k_N) \to (4-D)/F(0)$ and $\nu(k_N)$ becomes very large. As $k_N \to \infty$, $\nu(k_N) \to \nu_0$. The measure of the importance of viscous dissipation is $k_N{}^2\nu(k_N)$. For $k_N \to 0$ this behaves as

$$k_N^2\, \nu(k_N) \underset{k_N \to 0}{\sim} \left(k_N^2\right)^{1-(4-D)/6}, \tag{267}$$

so that, indeed as one expects, in the production region where the fluid is being mixed, viscous dissipation remains unimportant. As $k_N \to \infty$ this dissipation behaves as $k_N{}^2\nu_0$ and is the dominant feature of the deep dissipation region.

It is worthwhile emphasizing that the heart of these results is independent of the values of the physical parameters γ_0, ν_0 and k_0 which enter the bare lagrangian.

The large k behavior is governed essentially by the rapid convergence of the integrals encountered in perturbation theory. The small k behavior has any g, calculated from any g_0, approaching g_1 whose value is set by the details of the tensor structure of the $\vec{v}\cdot\nabla v_j$ vertex and the dimensions of space.

I want to close this section of the lecture by asking for more detail about the small wave number limit. For ease, I will assume that g is precisely equal to g_1 and derive consequences for the velocity-velocity correlation function $\Phi^{(2,0)}(k^2,\omega)$. Since $g = g(0) = g_1$, $\hat{g}(\tau) = g_1$ for all τ. Similarly

$$\frac{1}{\hat{\nu}(\tau)}\frac{d\hat{\nu}(\tau)}{d\tau} = -\frac{1}{1-B(g_1,\sigma)} = -1\Big/\left(1+\frac{g_1^2F(\sigma)}{D+2}\right) \tag{268}$$

$$= -\frac{D+2}{6}, \tag{269}$$

and

$$\hat{\nu}(\tau) = \nu\, e^{-(D+2)\tau/6} \tag{270}$$

or

$$\hat{\nu}(-\log\xi) = \nu\,\xi^{(D+2)/6}. \tag{271}$$

The solution to the renormalization group equation becomes

$$\Phi_R^{(2,0)}(\xi k^2,\omega,g_1,\nu,\sigma,k_N^2) = \xi^{\beta}\,\Phi_R^{(2,0)}(k^2,\omega,g_1,\xi^{\alpha}\nu,\xi\sigma,k_N^2) \tag{272}$$

with $\alpha = (D+2)/6$ and $\beta = -(D+2)/2$. The dimensions of $\Phi^{(2,0)}$ are

$$[\Phi^{(2,0)}] = [\gamma_0^2][G^{(2,0)}] = \omega\, k^{-(D+2)}, \tag{273}$$

which allows us to write

$$\Phi_R^{(2,0)}(k^2,\omega,g,\nu,\sigma,k_N^2) = \nu\left(k_N^2\right)^{-D/2} \times \psi\left(\frac{\omega}{\nu k_N^2}, k^2/k_N^2, g, \sigma\right) \tag{274}$$

where ψ is a dimensionless function of its arguments. The scaling rule (272) means ψ must be

$$\psi\left(\frac{\omega}{\nu k_N^2}, \frac{k^2}{k_N^2}, g_1, \sigma\right) = \left(k^2/k_N^2\right)^{\alpha+\beta} F\left(\frac{\omega}{\nu k_N^2}\left(\frac{k_N^2}{k^2}\right)^{\alpha}, \frac{\sigma k^2}{k_N^2}, g_1\right) \tag{275}$$

with F another dimensionless function.

The energy spectrum function $E(k)$ is related to $\Phi^{(2,0)}(k^2, \;)$ by (191) which means

$$E(k) \propto k^{D-1}\int d\omega\, \Phi^{(2,0)}(k^2,\omega) \tag{276}$$

or

$$E(k) \propto k^{D-1}\left(k^2\right)^{2\alpha+\beta}, \tag{277}$$

$$\propto k^{D-3-\rho} \quad \text{with } \rho = -(4-D)/3 \tag{278}$$

At D=3 in the case of stationary, isotropic turbulence we expect $E(k) \sim k^{-\rho}$, $\rho \approx -1/3$, for small k. For very large k the dissipative term $\nu_0\nabla^2 v_j$ dominates over the inertial transfer, so the linearized Navier-Stokes equation applies. In that very large k regime

$$E(k) \sim k^{D-3}\,\Gamma_m\left(k^2/k_0^2\right). \tag{279}$$

For the inertial range $k_0 \ll k \ll \eta^{-1}$, we must employ more of the renormalization group.

To interpolate between the known $k \to 0$ and $k \to \infty$ limits for $E(k)$, we will use the method outlined before for constructing general representations for the factors Z_ν, $\mathcal{Z}$, etc. which incorporate the effects of the interaction. We follow in our approach the work of Abarbanel, Bartels, Bronzan, and Sidhu (1975) and Frazer, Hoffman, Fulco and Sugar (1976) as adapted to homogeneous, isotropic, stationary turbulence by Abarbanel (1978b).

The idea is to establish differential equations for the dimensionless renormalization factors Z_ν, $\mathcal{Z}$, and ξ which respectively rescale the viscosity ν_0 to a renormalized viscosity

$$\nu = Z_\nu \nu_0 \tag{280}$$

the dimensionless expansion parameter g_0 to

$$g = \mathcal{Z} g_0 \tag{281}$$

and the velocity correlation function. Each of these functions depends on the dimensionless variables

$$g_0 = \left(\gamma_0 / \nu_0^{3/2}\right) \left(q_N^2\right)^{D-4/4}, \tag{282}$$

$$x_0 = \nu_0 q_N^2 / i\omega_N \tag{283}$$

and

$$\sigma = q_N^2 / k_0^2 \tag{284}$$

which enter the evaluation of $G^{(n,m)}(\vec{k}_i, \omega_i, \gamma_0, \nu_0, k_0)$ at an arbitrary point q_N, ω_N in wave number, frequency space.

The differential equations we seek will express Z_ν, $\mathfrak{z}$, and ζ in terms of g, $x=\nu q_N^2/i\omega_N$, and σ. In these differential equations, which taken together contain the same information as the renormalization group equation (242), will enter renormalization group functions like A and B above. Approximations to these functions by a perturbation series in g yield non-perturbative expressions for Z_ν, $\mathfrak{z}$, ζ in g or g_0. Our goal will be to give these expressions for Z_ν, $\mathfrak{z}$, and ζ as functions of $\vec{k}_i$, ω_i, γ_0, ν_0, and k_0 in appropriate dimensionless combinations. Then $G_{jl}^{(1,1)}$ and $G_{jl}^{(2,0)}$ will be determined. We will carry through the analysis for general Γ_M with the only approximation being the determination of the renormalization group functions in lowest order of g^2. This is called the one loop approximation as only the one loop diagrams like Figs. 10 and 11 are kept in their evaluation. The convergence of a series in g^2 for these functions is unlikely, but techniques now exist to determine the nature of the series and to perform a sum of the large order terms under some circumstances (Lipatov (1977), Brezin, le Guillou, and Zinn-Justin (1977)). We will limit ourselves to the one loop expression.

We want to determine the ω and k dependence of the scalar functions $\mathcal{G}^{(1,1)}$ and $\mathcal{G}^{(2,0)}$. So we will choose their normalizations at the point $k^2 = q_N^2$ and $\omega = i\omega_N$ in order to study both the q_N and ω_N variations. First we examine the scale factor Z_ν, which relates

$$\mathcal{G}^{(1,1)}(k^2, \omega, \gamma_0, \nu_0, k_0) = Z_1^{-1} \mathcal{G}_R^{(1,1)}(k^2, \omega, g, \nu, \sigma, q_N^2, \omega_N) . \qquad (285)$$

We will choose

$$\frac{\partial}{\partial \omega} \mathcal{G}_R^{(1,1)\,-1} \Big|_{\substack{k^2 = q_N^2 \\ \omega = i\omega_N}} = -i \qquad (286)$$

which makes $\mathcal{G}_R^{(1,1)}$ "close to" $\mathcal{G}^{(1,1)}$ for $\gamma_0 = 0$ when it is $-i\omega + \nu_0 k^2$. This allows us to determine $Z_1(g,\sigma,x)$ by

$$i \frac{\partial}{\partial \omega} \mathcal{G}^{(1,1)}(k^2, \omega, \gamma_0, \nu_0, k_0) \Big|_{\substack{k^2 = q_N^2 \\ \omega = i\omega_N}} = Z_1(g, \sigma, x) , \qquad (287)$$

and $\mathcal{G}^{(1,1)}$ is evaluated in a perturbation series in g_0 or by any other method.

With Z_1 in hand we can find Z_ν the rescaling parameter for the viscosity from

$$\mathcal{G}_R^{(1,1)}(k^2, \omega)^{-1} \Big|_{\substack{k^2 = q_N^2 \\ \omega = i\omega_N}} = Z_1^{-1} \left(\omega_N + \nu_0 q_N^2 Z_\nu \right) \qquad (288)$$

which means

$$Z_\nu(g, \sigma, x) = 1 - \frac{1}{\nu_0 q_N^2} \Sigma(\omega = i\omega_N, k^2 = q_N^2, \gamma_0, \nu_0, k_0) . \qquad (289)$$

In these formulae

$$g = \mathcal{Z} g_0 = \frac{Z_\gamma}{Z_\nu^{3/2}} g_0 , \qquad (290)$$

$$x = Z_\nu x_0 = \nu q_N^2 / i\omega_N , \qquad (291)$$

and

$$\sigma = q_N^2 / k_0^2 . \qquad (292)$$

Z_γ will be determined from the nor lization condition on $\Gamma_{n,1j}$ which we choose to be identical to our previous one. If we desired the k_i dependence of $\Gamma_{n,1j}$ we would have to normalize at $\vec{k}_i \neq 0$. Since $\vec{k}_i = 0$, we know that to lowest order in $\gamma_0{}^2$, $Z_\gamma = 1$.

Our goal will be to determine the function ζ given as

$$\mathcal{G}^{(2,0)}(k^2, \omega, \gamma_0, \nu_0, k_0)\Big|_{\substack{\omega = i\omega_N \\ k^2 = q_N^2}} = \frac{\Gamma_M(\sigma)}{-\omega_N^2 + \nu_0^2 q_N^4} \, \zeta(g, \sigma, x) . \qquad (293)$$

$\zeta(g, \sigma, x)$ carries all the information on how $\mathcal{G}^{(2,0)}(k^2, \omega)$ departs from its lowest order value —i.e., linearized Navier Stokes—$\Gamma_M(k^2/k_0{}^2)/(\omega^2 + \nu_0{}^2 k^4)$.

Now we wish to derive the differential equations for Z_ν, $\mathfrak{z}$, and ζ. (We'll use $Z_\gamma = 1$ or $\mathfrak{z} = Z_\nu{}^{-3/2}$ only after the general discussion.) We need the renormalization group functions

$$A_\omega = \omega_N \frac{\partial}{\partial \omega_N} g \Big|_{\gamma_0, \nu_0, k_0 \text{ fixed}} , \qquad (294)$$

$$A_q = q_N^2 \frac{\partial}{\partial q_N^2} g \Big|_{\gamma_0, \nu_0, k_0 \text{ fixed}} , \qquad (295)$$

and

$$A_0 = k_0^2 \frac{\partial}{\partial k_0^2} g \Big|_{\gamma_0, \nu_0 \text{ fixed}} \qquad (296)$$

To study Z_ν we also require

$$(B_\omega, B_g, B_0) = \left(\omega_N \frac{\partial}{\partial \omega_N}, g_N^2 \frac{\partial}{\partial g_N^2}, k_0^2 \frac{\partial}{\partial k_0^2}\right) \log Z_\nu \qquad (297)$$

with the same variables held fixed.

These functions A and B are to be determined as follows: in perturbation theory (or whatever method one prefers) find $\gamma(\gamma_0, \nu_0, k_0, \omega_N, q_N)$ and $\nu(\gamma_0, \nu_0, k_0, \omega_N, q_N)$ using the given normalization conditions. After performing the derivatives indicated, replace γ_0 and ν_0 by γ and ν consistent with the order of approximation used. This yields $A_\alpha(g,\sigma,x)$ and $B_\alpha(g,\sigma,x)$ $(\alpha = \omega, q, 0)$.

We turn to the chain rule to express general conditions on going over from γ_0, ν_0, k_0 to g, σ, x. This yields

$$\frac{A_\omega}{g} = \omega_N \frac{\partial}{\partial \omega_N} \log \mathfrak{z} \Big|_{\gamma_0, \nu_0, k_0 \text{ fixed}} \qquad (298)$$

$$= A_\omega \frac{\partial}{\partial g} \log \mathfrak{z}(g,\sigma,x) + (B_\omega - 1)\, x \frac{\partial}{\partial x} \log \mathfrak{z}(g,\sigma,x), \qquad (299)$$

$$\frac{A_g + \frac{4-D}{4} g}{g} = A_g \frac{\partial}{\partial g} \log \mathfrak{z} + (B_g + 1)\, x \frac{\partial}{\partial x} \log \mathfrak{z} + \sigma \frac{\partial}{\partial \sigma} \log \mathfrak{z} \qquad (300)$$

and $$\frac{A_0}{g} = A_0 \frac{\partial}{\partial g} \log \mathfrak{z} + B_0\, x \frac{\partial}{\partial x} \log \mathfrak{z} - \sigma \frac{\partial}{\partial \sigma} \log \mathfrak{z} \qquad (301)$$

Solving these equations for $\frac{\partial}{\partial g} \log \mathfrak{z}$ we find

$$\frac{\partial}{\partial g}\log \mathfrak{Z}(g,\sigma,x) = \left[\frac{\hat{A}(g,\sigma,x)}{g} + \frac{4-D}{4}\left(1-B_\omega(g,\sigma,x)\right)\right]\Big/\hat{A}(g,\sigma,x) \, , \tag{302}$$

where $$\hat{A}(g,\sigma,x) = (1-B_\omega)(A_0+A_g) + (1+B_0+B_g)A_\omega \, . \tag{303}$$

This differential equation is to be solved with the boundary condition $\mathfrak{Z}(0,\sigma,x) = 1$. This is the explicit expression that when the transfer of energy from mode to mode goes to zero (that is, the non-linearity is absent) the linear theory, which is the "coefficient" of $\mathfrak{Z}$, applies. The normalization to 1 at g=0 is due to our conditions (286) (287), and (241). We have then

$$\mathfrak{Z}(g,\sigma,x) = \exp\int_0^g \frac{du}{\hat{A}(u,\sigma,x)}\left[\frac{\hat{A}(u,\sigma,x)}{u} + \frac{4-D}{4}\left(1-B_\omega(u,\sigma,x)\right)\right]. \tag{304}$$

In a similar fashion we deduce

$$Z_\nu(g,\sigma,x) = \exp\int_0^g du\ \hat{B}(u,\sigma,x)\Big/\hat{A}(u,\sigma,x) \quad , \tag{305}$$

with
$$\hat{B} = (1-B_\omega)(B_0+B_g) + (1+B_0+B_g)B_\omega = B_0 + B_g + B_\omega \tag{306}$$

We can derive, but will not need, a similar equation for Z_1.

The equation for $\zeta(g,\sigma,x)$ follows in the same manner. By calculating $\mathcal{Y}^{(2,0)}$ in perturbation theory in γ_0, we can find the functions

$$(C_\omega, C_g, C_0) = \left(\omega_N\frac{\partial}{\partial\omega_N}, g_N^2\frac{\partial}{\partial g_N^2}, k_0^2\frac{\partial}{\partial k_0^2}\right)\log\zeta(\gamma_0,\nu_0,k_0,\omega_N,g_N). \quad (30\ \ )$$

Again noting that $\zeta(g,\sigma,x) = 1$ when g=0, we write

$$\mathcal{L}(g,\sigma,x) = \exp \int_0^g du \;\hat{C}(u,\sigma,x)\,/\,\hat{A}(u,\sigma,x) \quad , \tag{308}$$

with

$$\hat{C} = (1-B_\omega)(C_0+C_g) + (1+B_0+B_g)\,C_\omega \tag{309}$$

Our strategy will then be to solve for $g = \mathcal{Z}(g,\sigma,x)g_0$ and $x = Z_\nu(g,\sigma,x)x_0$ as functions of g_0 and x_0 using our knowledge of the A and B. Then we will express $\mathcal{L}(g,\sigma,x)$ as $\mathcal{L}(g(g_0,x_0,\sigma),\, x(g_0,x_0,\sigma),\, \sigma\,)$. Since $q_N{}^2$ and ω_N were arbitrary, we'll have determined the ω and k^2 dependence of $\mathcal{G}^{(2,0)}$ by noting

$$\mathcal{G}^{(2,0)}(k^2,\omega,\gamma_0,\nu_0,k_0) = \frac{\Gamma_m\,(k^2/k_0^2)}{\omega^2+\nu_0^2 k^4}\;\mathcal{L}\big(g(\bar{g}_0,\bar{x}_0,\bar{\sigma}),x(\bar{g}_0,\bar{x}_0,\bar{\sigma}),\bar{\sigma}\big) \tag{310}$$

where

$$\bar{g}_0 = \frac{\gamma_0}{\nu_0^{3/2}}\left(k^2\right)^{\frac{D-4}{4}}, \tag{311}$$

$$\bar{x}_0 = \frac{\nu_0 k^2}{\omega} \tag{312}$$

and

$$\bar{\sigma} = k^2/k_0^2 \,. \tag{313}$$

At this stage all reference to ω_N and q_N will have disappeared. (A more elegant presentation would have avoided their introduction.) Their role was really an auxiliary one, to emphasize the manner in which we probed the dependence of correlation functions on the <u>scale</u> of wave number or frequency.

To proceed we need the expressions for $\mathcal{G}^{(1,1)}$ and $\mathcal{G}^{(2,0)}$ in perturbation theory to lowest non-trivial order of $\gamma_0{}^2$. $\mathcal{G}^{(1,1)}$ has been given before, but we repeat it here

$$\mathcal{G}^{(1,1)}(k^2,\omega,\gamma_0,\nu_0,k_0)^{-1} = -i\omega+\nu_0 k^2 - \Sigma(k^2,\omega,\gamma_0,\nu_0,k_0) \tag{314}$$

$$\Sigma = -\frac{\gamma_0^2}{4(2\pi)^D}\int \frac{d^Dq\,\Gamma_m(q^2/k_0^2)}{2\nu_0 q^2}\,\frac{q_\ell\Delta_{\ell j}(k)q_j\left[k^2/q^2 - \frac{2\vec{k}\cdot(\vec{k}-\vec{q})}{(D-1)(\vec{k}-\vec{q})^2}\right]}{\left(-i\omega+\nu_0(q^2+(\vec{q}-\vec{k})^2)\right)} \tag{315}$$

For $\zeta(k^2,\omega,\gamma_0,\nu_0,k_0)$ we must evaluate the graphs in Fig. 12. This gives

$$\zeta(k^2,\omega,\gamma_0,\nu_0,k_0) = 1 + \frac{\Sigma(k^2,\omega)}{-i\omega+\nu_0 k^2} + \frac{\Sigma^*(k^2,\omega)}{i\omega+\nu_0 k^2} + \Sigma_2(k^2,\omega)/\Gamma_m(k^2/k_0^2)$$

with

$$\Sigma_2(k^2,\omega,\gamma_0,\nu_0,k_0) = \frac{\gamma_0^2}{8(2\pi)^D}\int\frac{d^Dq}{2\nu_0 q^2}\,q_\ell\Delta_{\ell j}(k)q_j\,\Gamma_m(q^2/k_0^2)\,\Gamma_m\left((\vec{q}-\vec{k})^2/k_0^2\right)\times$$

$$\left[\frac{(D-2)k^2}{D-1} + \frac{\vec{k}\cdot\vec{q}\;\vec{k}\cdot(2\vec{q}-\vec{k})}{(D-1)(\vec{k}-\vec{q})^2}\right]\left\{\frac{1}{q^2}\left[\frac{1}{i\omega+\nu_0(q^2+(\vec{q}-\vec{k})^2)}\,\frac{1}{-i\omega+\nu_0((\vec{k}-\vec{q})^2-\vec{q}^{\,2})} + \right.\right.$$

$$\left.\frac{1}{-i\omega+\nu_0((\vec{k}-\vec{q})^2+\vec{q}^{\,2})}\,\frac{1}{i\omega+\nu_0((\vec{k}-\vec{q})^2-\vec{q}^{\,2})}\right] + \frac{1}{(\vec{k}-\vec{q})^2}\left[\frac{1}{i\omega+\nu_0(q^2+(\vec{q}-\vec{k})^2)}\,\frac{1}{-i\omega+\nu_0(q^2-(\vec{q}-\vec{k})^2)}\right.$$

$$\left.\left.+\frac{1}{-i\omega+\nu_0((\vec{k}-\vec{q})^2+q^2)}\,\frac{1}{i\omega+\nu_0(q^2-(\vec{q}-\vec{k})^2)}\right]\right\}. \tag{316}$$

From these expressions we can find the A, B, and C functions we require. To the order in G^2 we have calculated we may write

$$\widetilde{A}(g,x,\sigma) = -\frac{(4-D)}{4}g + a(x,\sigma)g^3 \tag{317}$$

$$\widehat{B}(g,x,\sigma) = -b(x,\sigma)g^2 \tag{318}$$

$$B_\omega(g,x,\sigma) = b_\omega(x,\sigma)g^2 \tag{319}$$

and

$$\widetilde{C}(g,x,\sigma) = c(x,\sigma)g^2 \tag{320}$$

Integrating the equation for $\mathfrak{z}(g,x,\sigma)$ yields

$$\mathfrak{z}(g,x,\sigma) = \left(1 - g^2/g_1^2(x,\sigma)\right)^{\frac{1}{2} - \frac{4-D}{8} b_\omega(x,\sigma)/a(x,\sigma)} \tag{321}$$

where

$$g_1^2(x,\sigma) = (4-D)/4a(x,\sigma) \tag{322}$$

which is the zero of $\widehat{A}(g,x,\sigma)$ with $\frac{\partial\widehat{A}}{\partial g} > 0$. In the same fashion we learn

$$Z_\nu(g,x,\sigma) = \left(1 - g^2/g_1^2\right)^{-b/2a}, \tag{323}$$

and

$$\zeta(g,x,\sigma) = \left(1 - g^2/g_1^2\right)^{c/2a} \tag{324}$$

With our normalization condition on the fusion vertex we found $Z_\gamma = 1$ to the present order of accuracy. This means $\mathfrak{z} = Z_\nu^{-3/2}$ and

$$a = \frac{3b}{2} + \frac{(4-D)}{4} b_\omega \tag{325}$$

If we use the equations like (299)-(301) to study the formulae for $x\frac{\partial}{\partial x}\log \mathcal{Z}$ or $\sigma\frac{\partial}{\partial\sigma}\log\mathcal{Z}$, we see that each is a ratio of polynomials in g^2 with $\widetilde{A}(g,\sigma,x)$ as denominators. From (321), however, we find, for example,

$$x\frac{\partial}{\partial x}\log\mathcal{Z}(g,\sigma,x) = -\frac{(4-D)}{8}\,x\frac{\partial}{\partial x}\left(\frac{b_\omega(x,\sigma)}{a(x,\sigma)}\right)\log\left(1-g^2/g_1{}^2\right) + \\ +\left(\frac{1}{2}-\frac{4-D}{8}\frac{b_\omega}{a}\right)\left(1-g^2/g_1^2\right)^{-1} 2g^2/g_1{}^3 \;\; x\frac{\partial}{\partial x}g_1(x,\sigma). \tag{326}$$

The pole at $g^2=g_1{}^2$ here just reflects the zero of $\widetilde{A}(g,\sigma,x)$, but in the general formula for $x\frac{\partial}{\partial x}\log\mathcal{Z}$, there is no logarithmic factor. We conclude that b_ω/a must be independent of x. Similarly, we would infer that b_ω/a is independent of σ. The same arguments tell us that b/a and c/a are constants.

These rules are, in fact, only approximatèly obeyed at our order ($O(g^2)$) in determining $\widetilde{A}$, $\widetilde{B}$ and $\widehat{C}$. We may use this observation, however, to extract values for the ratios by conveniently choosing x and σ. By taking $\omega_N \to 0$ with $q_N{}^2$ and $k_0{}^2$ fixed; that is, $x\to\infty$ and then $k_0 \to 0$, we find that $b_\omega/a = 0$, so

$$a = 3b/2 \tag{327}$$

and the g, g_0 relation is

$$g^2 = \mathcal{Z}^2 g_0^2 = g_0^2\left(1-g^2/g_1^2\right)^{-1}, \tag{328}$$

$$\text{or}\quad 1-g^2/g_1^2 = \left(1+g_0^2/g_1^2\right)^{-1}. \tag{329}$$

It is clear from this that g^2 is bounded between O and

g_1^2 as g_0, the unrenormalized dimensionless expansion parameter, ranges in $0 \le g_0^2 < \infty$. In the same limit on ω_N and k_0 we find by numerical integration

$$\frac{c}{b} = 1.1496 \approx 8/7 \tag{330}$$

The implicit equation for $x(g_0^2, x_0, \sigma)$ reads

$$x = x_0 \left(1 + \frac{4 R_0^2}{4-D} \left(\frac{a}{c}\right) \sigma^{-\frac{D-4}{2}} c(x,\sigma) \right)^{1/3} \tag{331}$$

where we recall the Reynolds number based on the external scale k_0

$$R_0 = \frac{\gamma_0}{\nu_0^{3/2}} \left(k_0^2\right)^{\frac{D-4}{2}} \tag{332}$$

From $x(R_0^2, \nu_0 k^2/\omega , k^2/k_0^2)$ we have

$$\zeta(k^2, \omega, \gamma_0, \nu_0, k_0) = \left[x(R_0^2, \nu_0 k^2/\omega , k^2/k_0^2) \Big/ \left(\frac{\nu_0 k^2}{\omega}\right) \right]^{-8/7} , \tag{333}$$

which is the answer we are seeking. The difficult step here is the numerical evaluation of $x(R_0^2, \nu_0 k^2/\omega , k^2/k_0^2)$ from the implicit formula (331). That work is presently in progress.

We can, with no numerical work, verify that our construction does have the desired $k \to \infty$ and $k \to 0$ limits. As k^2 becomes very large, $\zeta \to 1$ or $x \to \nu_0 k^2/\omega$. A conservative estimate of "large" arises when we suppose $a(x,\sigma)$ is order unity (which is actually very conservative). Then when

$$R_0^2 \sqrt{\frac{k_0}{k}} \ll 1 \quad , \quad \zeta \simeq 1 \quad , \tag{334}$$

we surely have large k. This means $k \gg R_0^4 k_0^2$ which is enormous, when R_0 is sizeable. The decrease of $a(x, k^2/k_0^2)$ for large k will give a less outlandish estimate for "large" k.

When $k^2 \to 0$, $a(x, k^2/k_0^2)$ is finite and from the implicit equation for x we see that x is a function of only the combination $\omega^{-1}(k^2)^{1+(D-4)/6}$ in this limit. Since for small k^2, $G^{(2,0)}(k^2, \omega)$ is essentially

$$(\omega^2 + v_0^2 k^4)^{-1} \zeta\left(a(x,0), x\right), \tag{335}$$

we have for E(k)

$$E(k) \sim k^{D-1} \int_{-\infty}^{+\infty} d\omega \, G^{(2,0)}(k^2, \omega) \tag{336}$$

$$\sim k^{D-3-\rho} \quad , \quad \text{with} \quad \rho = -(4-D)/3 \, , \tag{337}$$

as before. With the proper $k^2 \to \infty$ and $k^2 \to 0$ limits, we can expect a sensible approximation to the intermediate range of k. For this the implicit equation for $x(R_0^2, v_0 k^2/\omega, k^2/k_0^2)$ must be analyzed numerically.

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APPENDIX A

Scattering From a Rough Surface

In this appendix we will cast the problem of scattering of scalar waves from a rough surface into a form similar to that of scalar wave propagation in a random medium. Our particular example will be that of an acoustic source located near a random surface, and we will consider the case where the wave amplitude is required to vanish on the surface. The physical situation for which this is important is that of an acoustic source near the ocean-atmosphere interface. Because the density of sea water and the speed of sound in sea water are both much greater than their values in the air, the ocean surface is essentially perfectly reflecting for acoustic waves.

We will consider, then, a sound source located above a surface $Z = \zeta(x,y,t)$ in a medium with constant sound speed. The medium will be taken to extend to $Z \rightarrow +\infty$. We are interested in the signal (sound pressure) $p(Z,\underset{\sim}{x},t)$ received at $Z,\underset{\sim}{x}=(x,y)$ at time t from a source at Z_0, $\underset{\sim}{x}_0$; $Z_0 \geqslant \zeta(\underset{\sim}{x},t)$. If the source has time dependence $s(t)$, the save equation for p reads

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) p(z,\underset{\sim}{x},t) = s(t)\,\delta(z-z_0)\,\delta^2(\underset{\sim}{x}-\underset{\sim}{x}_0)\,, \tag{A1}$$

and we wish to find the solution such that p vanishes on the surface

$$p(\zeta(\underset{\sim}{x},t),\underset{\sim}{x},t)=0, \tag{A2}$$

and at $Z^2 + \underset{\sim}{x}^2 \to \infty$.

We will treat only the problem where s(t) is monochromatic, $s(t) = e^{-i\omega t}$, and assume that the motions in $\zeta(\underset{\sim}{x},t)$ are slow compared to $e^{-i\omega t}$. This leads us to the Helmholtz equation ($k=\omega/c$)

$$(\nabla^2+k^2)\,\psi(z,\underset{\sim}{x}) = \delta(z-z_0)\,\delta^2(\underset{\sim}{x}-\underset{\sim}{x}_0) \tag{A3}$$

with the boundary conditions

$$\psi(\zeta(\underset{\sim}{x}),\underset{\sim}{x})=0 \tag{A4}$$

and

$$\psi(z,\underset{\sim}{x}) \to 0\ ,\quad z^2+\underset{\sim}{x}^2 \to \infty\ . \tag{A5}$$

The complication in this problem lies in the boundary condition (A4) on the random surface $Z=\zeta(\underset{\sim}{x})$. We propose to make a change of variables which will allow us to always satisfy the boundary conditions at the expense of complicating the wave equation (A3).

We go over to the coordinates

$$\xi = z-\zeta(\underset{\sim}{x}) \tag{A6}$$

and

$$\rho = \underset{\sim}{x} \tag{A7}$$

This maps the random surface to the plane $\xi=0$. In these coordinates the boundary conditions on the pressure field $P(\xi,\rho) = p(Z,\underset{\sim}{x})$ become

$$P(0, \rho) = 0 \tag{A8}$$

and $$P(\xi, \rho) \to 0 \quad \xi^2 + \rho^2 \to \infty \tag{A9}$$

The wave equation is transformed using the chain rule

$$\frac{\partial}{\partial z} P(\xi, \rho) = \frac{\partial}{\partial \xi} P(\xi, \rho) \tag{A10}$$

and $$\frac{\partial}{\partial x_j} P(\xi, \rho) = \frac{\partial}{\partial \rho_j} P(\xi, \rho) - \frac{\partial \zeta}{\partial \rho_j} \frac{\partial P}{\partial \xi}(\xi, \rho) . \quad j = 1, 2 \tag{A11}$$

This transforms (A3) into

$$\left[\left(\frac{\partial^2}{\partial \xi^2} + \nabla_\perp^2 + k^2 \right) - F(\xi, \rho) \right] P(\xi, \rho) =$$
$$\delta(\xi - (z_0 - \zeta(\rho_0))) \, \delta^2(\rho - \rho_0) , \tag{A12}$$

with $$\nabla_\perp = \left(\frac{\partial}{\partial \rho_1} , \frac{\partial}{\partial \rho_2} \right) \tag{A13}$$

the gradient operator in the ρ plane, and the <u>fluctuation operator</u>

$$F(\xi, \rho) = -(\nabla_\perp \zeta)^2 \frac{\partial^2}{\partial \xi^2} + \nabla_\perp^2 \zeta \frac{\partial}{\partial \xi}$$
$$+ 2(\nabla_\perp \zeta)_j \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \xi} . \tag{A14}$$

To solve the wave equation (A12) subject to the boundary conditions, we seek a Green function $G_0(\xi, \rho; \xi', \rho')$ satisfying

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial \xi^2} + k^2\right) G_0(\xi, \rho; \xi', \rho') = \delta(\xi - \xi')\,\delta^2(\rho - \rho') \qquad (A15)$$

and vanishing on the surface $\xi = 0$ and $\xi^2 + \rho^2 \to \infty$. This G_0 comes from putting an image source at $(-\xi', \rho')$ to match the source at (ξ', ρ') in (A15). We find

$$G_0(\xi, \rho; \xi', \rho') = \int \frac{d^3Q}{(2\pi)^3} \frac{e^{i\mathbf{Q}\cdot(\rho-\rho')}}{k^2 - \mathbf{Q}^2 - Q_3^2 + i\varepsilon} \left[e^{iQ_3(\xi-\xi')} - e^{iQ_3(\xi+\xi')} \right] \qquad (A16)$$

$$= -\frac{1}{4\pi}\left(\frac{e^{ikR_+}}{R_+} - \frac{e^{ikR_-}}{R_-} \right), \qquad (A17)$$

with

$$R_\pm^2 = \left((\xi \mp \xi')^2 + (\rho - \rho')^2 \right) \qquad (A18)$$

Since G_0 vanishes on the surface enclosing the source at $\xi = z_0 - \zeta(\rho_0)$ and ρ_0 and the receiver at $\xi = z - \zeta(\rho)$ and ρ, Green's theorem tells us that

$$P(\xi, \rho) = G_0(\xi, \rho; z_0 - \zeta(\rho_0), \rho_0) + \int_0^\infty d\xi' \int d^2\rho'\, G_0(\xi, \rho; \xi', \rho')\, F(\xi', \rho')\, P(\xi', \rho'). \qquad (A19)$$

The formal solution to this is

$$P = G_0 S + G_0 F P \qquad (A20)$$

or

$$P = (G_0^{-1} - F)^{-1} S, \qquad (A21)$$

with G_0^{-1} the operator $(\partial^2/\partial\xi^2 + \nabla_\perp^2 + k^2)$, F as above, and

$$S = \delta(\xi - (z_0 - \zeta(\rho_0)))\,\delta^2(\rho - \rho_0) \qquad (A22)$$

The task then is to invert the operator $G_0^{-1} - F$ with the definition (A16) or (A17) of G_0. If we call

$G^{-1} = G_0{}^{-1} - F$, then

$$p(z,\underset{\sim}{x}) = G(z-\xi(\underset{\sim}{x}),\underset{\sim}{x}\,;\, z_0-\xi(x_0),x_0) \qquad (A23)$$

is the answer to the problem posed. If we expand, as usual, in the fluctuation operator F and use the Green function G_0, then every term in the series or any approximation to the series will satisfy the boundary conditions.

Consider the term with F=0 which will be a good approximation when the gradients of ξ are small. Then

$$p(z,\underset{\sim}{x}) = G_0(z-\xi(\underset{\sim}{x}),\underset{\sim}{x}\,;\, z_0-\xi(\underset{\sim}{x}_0),\underset{\sim}{x}_0)$$

$$= \int \frac{d^3Q}{(2\pi)^3}\frac{e^{i\underset{\sim}{Q}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_0)}}{k^2-\underset{\sim}{Q}^2-Q_3^2+i\varepsilon}\Big[e^{iQ_3(z-z_0)}\,e^{-iQ_3(\xi(\underset{\sim}{x})-\xi(\underset{\sim}{x}_0))} -$$

$$- e^{iQ_3(z+z_0)}\,e^{-iQ_3(\xi(\underset{\sim}{x})+\xi(\underset{\sim}{x}_0))}\Big] \qquad (A24)$$

If ξ is a gaussian random surface with zero mean and correlation function

$$\langle \xi(\underset{\sim}{x})\,\xi(\underset{\sim}{y})\rangle = \Gamma(\underset{\sim}{x}-\underset{\sim}{y}) = \Gamma(\underset{\sim}{y}-\underset{\sim}{x}) \qquad (A25)$$

then the average pressure received at x, Z, is

$$\langle G_0(z-\xi(\underset{\sim}{x}),\underset{\sim}{x}\,;\, z_0-\xi(\underset{\sim}{x}_0),\underset{\sim}{x}_0)\rangle = \int \frac{d^3Q}{(2\pi)^3}\frac{e^{i\underset{\sim}{Q}\cdot(\underset{\sim}{x}-\underset{\sim}{x}_0)}}{k^2-\underset{\sim}{Q}^2-Q_3^2+i\varepsilon}\times$$

$$\left[e^{iQ_3(z-z_0) - Q_3^2(\Gamma(0)-\Gamma(\underset{\sim}{x}-\underset{\sim}{x}_0))} - e^{iQ_3(z+z_0) - Q_3^2(\Gamma(0)+\Gamma(\underset{\sim}{x}-\underset{\sim}{x}_0))} \right] \tag{A26}$$

Introducing the parameter λ via

$$(k^2 - \underset{\sim}{Q}^2 - Q_3^2 + i\varepsilon)^{-1} = -i \int_0^\infty d\lambda \; \exp i\lambda(k^2 - \underset{\sim}{Q}^2 - Q_3^2 + i\varepsilon) \tag{A27}$$

we can perform the required gaussian integrals to arrive at

$$\begin{aligned} &\langle G_0(z-\xi(\underset{\sim}{x}), \underset{\sim}{x}; z_0 - \xi(\underset{\sim}{x}_0), \underset{\sim}{x}_0)\rangle = \\ &\quad -\frac{1}{8\pi^{3/2}} \int_0^\infty \frac{d\lambda}{\lambda} \, e^{i\lambda k^2 + i(\underset{\sim}{x}-\underset{\sim}{x}_0)^2/4\lambda} \left\{ \frac{e^{-(z-z_0)^2/4\gamma_-}}{\sqrt{\gamma_-}} \right. \\ &\qquad \left. - \frac{e^{-(z+z_0)^2/4\gamma_+}}{\sqrt{\gamma_+}} \right\} , \end{aligned} \tag{A28}$$

with $\gamma_\pm = \Gamma(0) \pm \Gamma(\underset{\sim}{x}-\underset{\sim}{x}_0) + i\lambda$. (A29)

For k large, this integral is dominated by $\lambda \sim ((\underset{\sim}{x}-\underset{\sim}{x}_0)^2)^{1/2}/k$, so we may neglect λ in $\gamma_\pm$ to write

$$\begin{aligned} &\langle G_0(z-\xi(\underset{\sim}{x}), \underset{\sim}{x}; z_0 - \xi(\underset{\sim}{x}_0), \underset{\sim}{x}_0)\rangle \approx \\ &\quad -\sqrt{\frac{2i}{kR}} \, \frac{e^{ikR}}{8\pi} \left\{ \frac{e^{-(z-z_0)^2/4\gamma_-'}}{\sqrt{\gamma_-'}} - \frac{e^{-(z+z_0)^2/4\gamma_+'}}{\sqrt{\gamma_+'}} \right\} \end{aligned} \tag{A30}$$

with $\gamma_\pm' = \Gamma(0) \pm \Gamma(\underset{\sim}{x}-\underset{\sim}{x}_0)$; $R^2 = (\underset{\sim}{x}-\underset{\sim}{x}_0)^2$.

Now to study more then the leading term in F, we must turn to our machinery developed for studying operators like $G_0^{-1} - F$ when F fluctuates. This is just the problem of a scalar wave in a random medium characterized by F. The only new twists here over and above the scalar wave propagation considered before are the facts that (1) the G_0 reflects the boundary condition at $\xi=0$ and (2) the source and receiver also fluctuation. This last feature is more or less harmless since going over to wave number space puts the source and receiver factors into the exponential at the outset, so averages over the fluctuations of $\xi(x)$ can be done as in going from (A24) to (A26).

APPENDIX B

Using the Fluctuation-Dissipation Theorem in Turbulence

In the paper by Forster, Nelson and Stephen (1977) an application is made of the "fluctuation-dissipation" theorem (see Orzag (1974), Sec. 5.3 and Ma (1976), Sec. XI.2) to the theory of turbulent flow. Here I want to review and comment on their results.

The discussion follows Ma (1976). We begin with a conservative system of variables $Q_j(t)$ which develop in time according to some law

$$\frac{\partial}{\partial t} Q_j(t) = w_j\left(Q_j(t)\right) \tag{B1}$$

given by hamilton's equations. $Q_j(t)$ includes the co-ordinates and their conjugate momenta, so the hamiltonian will be written $H(Q_j)$. Dissipation is a result of placing the original system into contact with a thermal reservoir or of averaging over some subset of variables of the whole system and considering the dynamics of the remaining variables with a parametrization of the averaged subset in terms of temperature, etc.

If we imagine, then, that the variables $Q_j(t)$ are the remaining variables after an average over some others, then we may write a Langevin equation for the motion of the Q_j

$$\frac{\partial}{\partial t} Q_j = w_j(Q_j) - \Gamma_j \frac{\partial H}{\partial Q_j} + \xi_j(t) \tag{B2}$$

where Γ_j is a damping coefficient which drives the system of Q_j's to an equilibrium state and $\zeta_j(t)$ describes the spontaneous thermal fluctuations associated with this dissipative process. It is essential to keep in mind that the dissipation coefficients Γ_j and the random forcing come from the same source: the averaged coordinates representing "the rest of the system"—the part whose dynamics we do not explore. This formulation allows a possible microscopic discussion of the origin and properties of the viscous term in the Navier-Stokes equation and will allow a connection between the damping coefficient, proportional to ν_0, and the molecular fluctuation forces. In fluid dynamics the averaging is clearly that which takes place in the definition of macroscopic fluid flow (see Batchelor (1970), Section 1.2).

The properties of the fluctuating forces ζ_j are taken to be those of a gaussian with mean zero and

$$\langle \zeta_j(t)\zeta_\ell(t')\rangle = 2 D_j \delta_{j\ell}\,\delta(t-t') \tag{B3}$$

The system of Q_j's now satisfies a Fokker-Planck equation for the probability distribution $P[Q_j,t]$. This is

$$\frac{\partial}{\partial t}P + \sum_\ell \frac{\partial}{\partial \varphi_\ell} J_\ell = 0 \tag{B4}$$

with the probability current

$$J_\ell = w_\ell P - \Gamma_\ell \frac{\partial H}{\partial \varphi_\ell} - D_\ell \frac{\partial P}{\partial \varphi_\ell} \tag{B5}$$

The derivation of such Fokker-Planck Equations is discussed by Wang and Uhlenbeck (1945).

The Q_j will relax to an equilibrium state where

$$\frac{\partial P}{\partial t} = 0 \tag{B6}$$

and in this state

$$P(Q_j) = \exp - \beta H(Q_j) = \exp - H / k_B T \tag{B7}$$

Namely the Boltzmann distribution determines the distribution of the variables Q_j in equilibrium. For (B.7) to occur it is necessary that

$$\Gamma_j / \beta = D_j \tag{B8}$$

This is the "fluctuation-dissipation" connection. It relates the dissipation rate Γ_j to the scale of fluctuations. It is also called the Einstein relation since a similar connection was made by him in the theory of Brownian motion. An early derivation is given by Kirkwood (1945).

To make contact with the Navier-Stokes equation we need to add spatial dependence to (B.2), so we write

$$\frac{\partial}{\partial t} Q_j(\vec{k},t) = w_j(Q_\ell(\vec{k},t)) - \Gamma_j(\vec{k}) \frac{\partial H(Q_\ell(\vec{k},t))}{\partial Q_j^*(\vec{k},t)} + \zeta_j(\vec{k},t) \tag{B9}$$

where the fluctuation force satisfies

$$\langle \zeta_j(\vec{k},t) \rangle = 0$$

$$\langle \zeta_j(\vec{k},t) \zeta_\ell(\vec{q},\tau) \rangle = \delta_{j\ell} D_j(\vec{k}) \delta^3(\vec{q}+\vec{k}) \delta(t-\tau) \tag{B10}$$

and

$$D_j(\vec{k}) = \Gamma_j(\vec{k}) k_B T \tag{B11}$$

to insure that the Boltzmann law holds in equilibrium. If we interpret $Q_j(\vec{k},t)$ as the microscopic velocity $v_j(\vec{k},t)$, we can ask when (B.9) gives us the Navier-Stokes equations. The viscous term in the Navier-Stokes equation $\nu_0 \nabla^2 v_j(\vec{x},t)$, will come from the $\Gamma_j(\vec{k})\, \partial H/\partial v_j(\vec{k},t)^*$. For H it is natural to choose

$$H = \int d^D k \left[\frac{1}{2} \rho \, |\vec{v}(\vec{k},t)|^2 + \cdots \right] , \qquad (B12)$$

and this is implicitly done by Forster, Nelson, and Stephen. Then we have

$$\nu_0 k^2 = \rho \, \Gamma(k^2) \qquad (B13)$$

and

$$D(k^2) = \frac{\nu_0 k^2}{\rho} k_B T \qquad (B14)$$

for the correlation function of the random molecular forces associated with the dissipation process.

With an appropriate choice of other terms in H and in the w_j we can construct

$$\frac{\partial}{\partial t} v_j(\vec{x},t) + \vec{v}\cdot\nabla v_j = \nu_0 \nabla^2 v_j + \zeta_j(\vec{x},t) - \frac{1}{\rho} \nabla_j p \qquad (B15)$$

which is familiar. It is very important to note at this point that the fluctuation—dissipation connection (B.14) or (B.11) is only able to relate aspects of the thermal motion which lies at the origin of the dissipation process. That, of course, is physically sensible since the molecular fluctuations and viscous dissipation in the fluid are parts of the same process.

What has all this to do with turbulent flow? If one attributes turbulence to the thermal fluctuations of the molecular motion of a fluid emersed in a heat bath, then I suppose (B.15) with the fluctuation-dissipation connection given in (B.14) is the equation one wishes to study. It seems to me very unlikely that turbulence is due solely to the thermal fluctuations of molecules. Just some thought about grid turbulence, or turbulent motion in the atmosphere, or internal waves in the ocean ought to make that clear. In the Langevin type of equation (B.2) or (B.9) we are trying to describe a system in which the slowing down of motion (think of Brownian motion) is due to precisely the same physical process which is stimulating the motion—in Brownian motion, the molecular fluctuations stimulate the motion of the "slow" or "big" system and collisions transfer the motion back to the molecules. Turbulent flow is quite different. The turbulent motion is stimulated by some agent "external" to the flow: a grid, a rough surface, wind interacting with the ocean surface, etc. The damping by viscosity is quite independent of the stimulating force. So no relation such as (B.8) or (B.14) should hold between ν_0 and the correlation function of the external force $f_j(x,t)$ which mixes the fluid and stimulates the motion of the many instabilities of the fluid which gives a fluid motion we call turbulence.

Nonetheless, there is a kind of balance equation for stationary turbulence which was given by Novikov (1964) and expresses the idea that the net dissipated energy/unit volume, $\mathcal{E}$, is related to the net input through the external forces. In the notation used in the text above, Novikov shows

$$\mathcal{E} = \frac{\gamma_0^2}{4}\,\frac{D-1}{2}\int\frac{d^D k}{(2\pi)^D}\;\Gamma_M\left(k^2/k_0^2\right) \tag{B16}$$

for a gaussian random force. This only restricts Γ_M to be such that the integral in (B.16) exists, but certainly does not require it to vanish at $k^2 \to 0$ as a glance at (B.14) might imply.

So when we force the fluid in a manner physically independent of the dissipation mechanism in the fluid, we conclude there is no requirement that $\Gamma_M(0) = 0$. Indeed, it seems somewhat unnatural. Combining these observations with those in the text about $\Gamma_M(0) \neq 0$, we can feel confident in that conclusion.