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The Dynamical Effects of Instantons

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ABSTRACT

The generating functional for SU(2) quantum-chromodynamics has a representation as a classical ensemble of interacting instantons. It is shown that the instantons do not directly contribute to quark confinement but only to quark mass renormalization. A perimeter type law for the phase factor of a large quark loop is derived for both dilute and dense ensemble configurations. The dilute ensemble is studied by an assumed cluster decomposition. The dense ensemble is studied by Monte Carlo computer simulation.

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I. INTRODUCTION

There is great interest in instanton solutions to gauge theories.[1-8] One of the major reasons for this interest is the hope that such solutions could lead to an understanding of the confinement mechanism in quantum-chromodynamics. Ever since the discovery of the instanton solution by Belavin, et al.[1] and the subsequent demonstration of how instanton solutions in three-dimensional QED lead to confinement,[2] there has been considerable hope and speculation that the same mechanism might also work for four-dimensional QCD. It is known that topological singularities like instantons contribute substantially to the dynamics of a system. One of the best examples is the role of vortices in the two-dimensional XY model.[9] The understanding of the dynamical effects of instantons is an important problem for QCD.

In this paper, we study the interactions of quarks and instantons in the SU(2) gauge theory. In particular, we calculate the expectation value for a large quark loop in the presence of an ensemble of interacting instantons. We show that this factor decreases exponentially with the perimeter and not with the area of the loop. This is demonstrated for both dilute and dense configurations of instantons under the assumption that the long range instanton-instanton forces can be neglected. Instantons do not directly contribute to quark confinement but only to quark mass renormalization.

If one starts with the generating functional of SU(2) QCD in Euclidean space which is given by

$$Z_{QCD} = \int \mathscr{D}A_\mu \mathscr{D}\psi \; \mathscr{D}\bar{\psi} \; \exp\left\{- S(A, \psi, \psi)\right\} , \tag{I.1}$$

where $S(A, \psi, \psi)$ is the SU(2) QCD action for gauge fields and fermions, one can show that the fermion degrees of freedom can be integrated out to obtain a sum over all possible closed quark paths:[10]

$$Z_{QCD} = \sum_{\text{paths } c} \langle \chi(c) \rangle , \tag{I.2}$$

where

$$\langle \chi(c) \rangle = \int \mathscr{D}A_\mu \frac{1}{2} \text{ tr } P \exp\left\{ i g \oint_c dx^\mu A^a_\mu \frac{\sigma^a}{2}\right\} \exp\left\{-S(A)\right\} . \tag{I.3}$$

$S(A)$ is the QCD action for gauge fields only, σ^a are the usual Pauli matrices, and P represents a path ordering of the line intergral.

But it has also been shown that the gauge field integration has a very rich vacuum and topological structure.[3-5] One way to account for this structure[4] is to sum over sectors of gauge field integration restricted to a specific winding number,

$$Z_{YM} = \int \mathscr{D}A_\mu \exp\left\{-S(A)\right\} = \sum_{n=-\infty}^{\infty} \int \left[\mathscr{D}A_\mu\right]_n \exp\left\{-S(A)\right\}, \tag{I.4}$$

where

$$n = \frac{g^2}{64\pi^2} \int d^4x \; \epsilon_{\mu\nu\lambda\sigma} \; F^a_{\mu\nu} \; F^a_{\lambda\sigma} . \tag{I.5}$$

We, of course, are choosing the $\theta = 0$ vacuum which is the relevant one for strong interactions (see however reference 11). A method to insure restricting the gauge field integration is to perturb about classical solutions constrained to a set winding number such that the perturbation does not alter the restriction.[3] Within a particular winding number sector, one could sum over contributions from instanton-anti-instanton pairs. However, to avoid overcounting, one would have to truncate the perturbation series in the coupling constant g. We choose not to truncate perturbation theory in the gauge field sector and thereby consider only expansions about instantons. However, including a summation over instanton-anti-instanton pairs for a dilute gas would not change the perimeter law for a large quark loop.

Classical instanton solutions do exist[6,7] and have independent parameters which crudely represent the positions x_i sizes λ_i and the isospin rotations R_i of n individual instantons. When quantizations are performed about these solutions, these independent parameters become quantum variables, known as collective coordinates,[12] to be integrated over. In this manner, the generating functional becomes a classical grand canonical ensemble,

$$Z_{YM} = \sum_{n=-\infty}^{\infty} \frac{1}{|n|!} \int \prod_{i=1}^{|n|} d^4x_i \frac{d\lambda_i}{\lambda_i^5} dR_i \exp\left\{K^n(x_i, \lambda_i, R_i, g, \mu)\right\}, \tag{I.6}$$

where μ is the subtraction point. The symbol K is used by analogy with the grand canonical Hamiltonian of finite-temperature many-body theory. Although one isospin integration corresponds to a redundant overall isospin rotation, we leave it in for ease of notation. If we assume that the effective coupling g_{eff} is small, then the contribution of a single quark loop can be approximated by the phase factor evaluated in the presence of the classical n instanton solution $A_\mu^{a\,cl}$.

$$<\chi(c)> \cong <\chi(c)>^{cl} = \sum_{n=-\infty}^{\infty} \frac{1}{|n|!} <\chi(c)>_n^{cl} \; , \tag{I.7}$$

where

$$<\chi(c)>_n^{cl} = \int \prod_{i=1}^{|n|} d^4x_i \frac{d\lambda_i}{\lambda_i^5} dR_i \frac{1}{2} \operatorname{tr} P \exp\left\{ i g_{eff} \oint_c dx^4 A_\mu^{a\,cl} \frac{\sigma^a}{2} \right\} \exp\{K^n\} . \tag{I.8}$$

Even though it is unlikely that the effective coupling for a large quark loop is small, a calculation of the expectation value $<\chi(c)>^{cl}$ could reveal whether the instantons alone could cause confinement. If they could not confine quarks without an infinite number of gluon exchanges, then the confinement mechanism could be seen to come primarily from strong coupling or other field configurations rather than instantons.

In this paper, we compute the expectation value of the classical phase factor in the presence of dilute and dense ensembles of instantons. The dilute gas is studied by the assumption of cluster decomposition. Unfortunately, cluster decomposition cannot be proven because the long range interactions between instantons (due to gluon exchange) are unknown. However, the short-distance interactions of instantons are better understood because the short distance behavior of the theory is known through the renormalization group. Therefore, as long as long range interactions can be neglected we are able to study ensembles that are sufficiently dense such that their short-distance interactions dominate. Using Monte Carle techniques borrowed from the theory of liquids,[13] we calculate the expectation value. The result for both dense and dilute systems is that an ensemble of instantons gives rise to exponential decrease with the perimeter P and not with the area A.

$$< \chi(c)>^{cl} \underset{P,A\to\infty}{\sim} \exp\{-aP\} \;. \tag{I.9}$$

This paper is organized as follows. Section II, describes how the renormalization group affects the instanton-instanton interaction. In Section III, we calculate the phase factor in the presence of one instanton. The dilute gas is studied in Section IV, and the dense ensemble is studied in Section V. Finally, we record some comments and conclusions in Section VI.

II. THE INSTANTON-INSTANTON INTERACTION

In this section, we study the interactions between instantons. Consider the n instanton potential energy function K^n. Its behavior is clearly demonstrated by use of the renormalization group. First of all, we observe that K^n is independent of the subtraction point μ. Insisting that $\partial K^n/\partial\mu \big|_{x_i, \lambda_i, R_i, g_{bare}\ \text{fixed}} = 0$ leads to the renormalization group equation

$$\left[x_i^\mu \frac{\partial}{\partial x_i^\mu} + \lambda_i \frac{\partial}{\partial \lambda_i} + \beta(g)\frac{\partial}{\partial g}\right] K^n(x_i, \lambda_i, R_i; g, \mu) = 0 \,. \tag{II.1}$$

If we now consider various rescalings of the parameters we observe several obvious but important facts. First, the interaction of instantons at large distances is controlled by strong coupling. Second, the potential for large instantons is also controlled by strong coupling. However, the potential for small sizes is controlled by weak coupling. And finally, the short distance interaction of instantons is also controlled by weak coupling.

In particular, if we consider the potential for two instantons, we have

$$K^2(x_1, x_2, R_i, \lambda_i; g, \mu) = K^2(y, \lambda_i, R_i; g, \mu) \ , \tag{II.2}$$

where $y = |x_1 - x_2|$ so that

$$\left[y \frac{\partial}{\partial y} + \lambda_i \frac{\partial}{\partial \lambda_i} + \beta(g) \frac{\partial}{\partial g} \right] K^2 = 0 \ , \tag{II.3}$$

implies

$$\begin{aligned} K^2(s\,y, \lambda_i, R_i; g, \mu) &= K^2\left(y, \frac{\lambda_i}{s}, R_i; \bar{g}\ (1/s), s\mu \right) \\ &\underset{s \to 0}{\sim} -2\, \frac{8\pi^2}{g^2(1/s)} \sim \frac{22}{3} \ \ln s \ , \end{aligned} \tag{II.4}$$

where $\bar{g}\ (1/s)$ is the running coupling. So the instanton interaction at short distances is logarithmically repulsive. If one examines the 'tHooft solution,[6] this result does not seem surprising. When an instanton has the same position as another instanton, the winding number changes abruptly by -1, so if we restrict ourselves to a specific winding number sector, the short-distance repulsion is merely giving zero probability for instantons to have the same position. Note that these same considerations for the n instanton potential would show that the ensemble could not collapse to a small region due to a short-distance repulsion.

III. THE QUARK-INSTANTON INTERACTION

In this section, we examine the quark-instanton interaction by calculating the phase factor

$$\chi = \frac{1}{2} \operatorname{tr} P \exp \left\{ i g \oint_C dx^\mu A^a_\mu \frac{\sigma^a}{2} \right\} . \tag{III.1}$$

In the presence of a single instanton, we will use both the Belavin, et al. instanton solution[1] and the 'tHooft anti-instanton solution[6] which are respectively given by:

$$A^a_\mu (x) = \frac{1}{g} \quad \eta^a_{\mu\alpha} \frac{2(x - x_1)^\alpha}{(x - x_1)^2 + \lambda^2} , \tag{III.2}$$

and

$$A^a_\mu (x) = \frac{1}{g} \quad \eta^a_{\mu\alpha} \frac{2(x - x_1)^\alpha \lambda^2}{(x - x_1)^2 [(x - x_1)^2 + \lambda^2]} , \tag{III.3}$$

where the imbedding matrix $\eta^a_{\alpha\beta}$ is defined by

$$\begin{aligned} \eta^a_{\alpha\beta} &= - \eta^a_{\beta\alpha} \\ \eta^a_{i4} &= \delta_{ai} \\ \eta^a_{ij} &= \epsilon_{aij} (i, j = 1, 2, 3) . \end{aligned} \tag{III.4}$$

Evaluation of χ will require care at infinity for the Belavin solution and care at the origin for the 'tHooft solution.

Let us parameterize the quark path by $x_\mu(t)$ where $t_o < t < T$ and define

$$\vec{A}(t) = g\,\vec{A}_\mu(x(t))\,\frac{dx^\mu(t)}{dt}\;, \tag{III.5}$$

and

$$\begin{aligned} U(t) &= T\exp\left\{\frac{i}{2}\int_{t_o}^{t} dt'\,\vec{A}(t')\cdot\vec{\sigma}\right\} \\ &= 1 + \sum_{m=1}^{\infty}\left(\frac{i}{2}\right)^m \frac{1}{m!}\int_{t_o}^{t} dt_1\ldots dt_m\, T\left\{\vec{A}(t_1)\cdot\vec{\sigma}\ldots\vec{A}(t_m)\cdot\vec{\sigma}\right\}, \end{aligned} \tag{III.6}$$

where the T ordering is defined by placing the $\vec{A}(t)\cdot\vec{\sigma}$ with the lowest value of t to the right, then the one with the next lowest value and so on. Then

$$\chi = \frac{1}{2}\,\mathrm{tr}\;U(T) \tag{III.7}$$

U(T) is in general difficult to calculate except numerically. However it can be calculated analytically in a few special cases which will indicate the general behavior.

Consider an instanton a distance b away from a straight line path as shown in Fig. 1. Choosing $x_\mu(t) = (t,0,0,0)$ and $x_1 = (0,-b,0,0)$, we find that $A_1 = A_2 = 0$ and the problem is reduced to an Abelien one. We can do the integral directly and obtain

$$U(t) = \exp\left\{ i\sigma_3\left[\frac{b}{\sqrt{b^2+\lambda^2}}\,\tan^{-1}\frac{t'}{\sqrt{b^2+\lambda^2}}\right]_{t_0}^{t}\right\}\;. \tag{III.8}$$

For a Belavin solution, and

$$U(t) = \exp\left\{ -i\sigma_3 \left[\tan^{-1}\frac{t'}{b} - \frac{b}{\sqrt{b^2+\lambda^2}} \tan^{-1}\frac{t'}{\sqrt{b^2+\lambda^2}} \right]_{t_0}^{t} \right\} , \tag{III.9}$$

for a 'tHooft solution. If we close the contour at infinity, the 'tHooft solution gives no additional contribution, but the Belavin solution gives an additional contribution of π to the integral. Both solutions give the result

$$\chi = \cos\left[\pi\left(1 - \frac{b}{\sqrt{b \quad + \lambda^2}}\right)\right] , \tag{III.10}$$

which has the behavior shown in Fig. 2. Likewise, the case of an instanton in the plane of a circular ring reduces to the Abelein case. Integration over the ring gives

$$\chi = -\cos\left\{\pi\left[1 + \frac{4R^2\lambda^2}{(R^2-\lambda^2-x_1^2)^2}\right]^{-1/2}\right\} , \tag{III.11}$$

which is plotted in Fig. 3. Note that if we let the radius R go to infinity, keeping the shortest distance to the quark path $b = \left| |x_1| - R \right|$ and the size λ fixed, we obtain Eq. (III.10).

Finally, we calculate the phase factor for a circular quark loop and an instanton on the axis of the loop at arbitrary time as shown in Fig. 4. First, we note that U(t) obeys a differential equation

$$\frac{d}{dt} U(t) = \frac{i}{2} \vec{A}(t) \cdot \vec{\sigma} \; U(t) . \tag{III.12}$$

Now we write U(t) in terms of Euler angles.

$$U(t) = \begin{pmatrix} e^{i(\psi+\phi)/2} \cos\theta/2 & i e^{i(\psi-\phi)/2} \sin\theta/2 \\ i e^{-i(\psi-\phi)/2} \sin\theta/2 & e^{-i(\psi+\phi)/2} \cos\theta/2 \end{pmatrix} \tag{III.13}$$

and Eq. (III.12) becomes

$$\begin{aligned} A_1 &= \frac{d\theta}{dt} \cos\psi + \frac{d\phi}{dt} \sin\theta \sin\psi \\ A_2 &= -\frac{d\theta}{dt} \sin\psi + \frac{d\phi}{dt} \sin\theta \cos\psi \\ A_3 &= \frac{d\phi}{dt} \cos\theta + \frac{d\psi}{dt} \quad . \end{aligned} \tag{III.14}$$

Observe that Eqs. (III.14) are the equations of motion of a rigid body with angular velocity $\vec{A}$.

If we let A^a_μ be a 'tHooft instanton we get

$$\vec{A}(t) = [a \sin(t+\delta), \ -a\cos(t+\delta), c] \ , \tag{III.15}$$

where

$$a = \frac{2R\lambda^2 |x_1|}{(R^2 + x_1^2)(R^2+\lambda^2+x_1^2)}$$

$$\delta = -\tan^{-1} x_1^3/x_1^4 \tag{III.16}$$

$$c = -\frac{2R^2\lambda^2}{(R^2+x_1^2)(R^2+\lambda^2+x_1^2)} \quad .$$

In this case, $\vec{A}$ is the same angular velocity as a torque-free rigid body.

The solution is

$$\theta = \tan^{-1}\left(-\frac{a}{c+1}\right)$$

$$\psi = -t \tag{III.17}$$

$$\phi = -\frac{a}{\sin\theta}\,t\ .$$

We want U(0) to be the unit martix, so we rotate by $-\theta$ to obtain

$$U(t)=\begin{pmatrix} e^{i(\chi+\phi)/2}\cos\theta/2 & ie^{i(\psi-\phi)/2}\sin\theta/2 \\ ie^{-i(\psi-\phi)/2}\sin\theta/2 & e^{-i(\psi+\phi)/2}\cos\theta/2 \end{pmatrix}\begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}, \tag{III.18}$$

and

$$\chi = [\cos\psi/2\ \cos\phi/2 - \sin\psi/2\ \sin\phi/2\ \cos\theta/2]\ t = 2\pi$$

$$= -\cos\left[\pi\sqrt{a^2+(c+1)^2}\,\right] \tag{III.19}$$

$$= -\cos\left\{\pi\left[1-\frac{4R^2\lambda^2}{(R^2+\lambda^2+x_1^2)^2}\right]^{1/2}\right\}\ .$$

This behavior is plotted in Fig. 5.

What can we now conclude about the general behavior of the quark-instanton interaction? From Figs. 2 and 3, we see that the greatest phase variation occurs when a quark path intersects an instanton and that the range of the variation is of the order of the size of the instanton. This is also evident in the large phase variation for instantons of the size of a loop within the same distance of the loop. However, instantons very much larger than the size of a loop do not cause large variation. Consequently, the phase factor for a general loop with a radius very much larger than the instanton size is given approximately by Eq. (III.9) and Fig. 2.

Instantons cause phase variation for a large quark loop only if they are within range of the quark path. The range is of the order of the instanton size.

IV. THE DILUTE GAS

In this section, we study the low density sector of the instanton ensemble. We use the dilute gas approximation and cluster decomposition. Unfortunately, we do not know the long range interaction of instantons because it is a strong coupling problem. In addition, there is no knowledge of the contributions of large instantons. Consequently, to use the dilute gas approximation requires an unprovable assumption of cutoff of the long range and large size behavior. We show that this assumption leads directly to the perimeter law Eq. (I.9).

First of all, let's consider the clustering properties of the phase factor itself. The phase factor χ^n does not have simple clustering properties in the presence of n instantons except that as one instanton goes very far from the quark loop

$$\chi^n(x_1 \ldots x_n) \xrightarrow[|x_n| \to \infty]{} \chi^{n-1}(x_1 \ldots x_{n-1}) \,. \tag{IV.1}$$

This is a property of all presently known solutions. However, the phase matrix $U = U(T)$ which is related to χ by Eq. (III.7), has more useful cluster properties. Consider a large circular loop (with a radius $R >> \lambda_1$) as shown in Fig. 4 and let $\phi_i = \tan^{-1}(x_i^2/x_i^1)$. Then the phase matrix in the presence of two instantons has the property.

$$U^2(x_1, x_2) \xrightarrow[|x_1 - x_2| \to \infty]{} \Phi\, U^1(x_1)\, U^1(x_2) \;, \tag{IV.2}$$

where Φ represents ordering the U matrices with the smallest value of ϕ to the right. Each $U^1(x_i)$ is the phase matrix for the loop in the presence of a single instanton. It is given by

$$U^1(x_i) = \exp\left\{ i\vec{\sigma}\cdot\hat{n}_i\, \pi \left(1 - \frac{b_i}{\sqrt{b_i^2 + \lambda_i^2}}\right)\right\} , \tag{IV.3}$$

where

$$b_i = \left[\left(R - \sqrt{(x_i^1)^2 + (x_i^2)^2}\right)^2 + (x_i^3)^2 + (x_i^4)^2\right]^{1/2} , \tag{IV.4}$$

and $\hat{n}_i$ is a unit vector determined by the embedding matrix $\eta^a_{\alpha\beta}$, the path, and the instanton position. Note that the Φ ordering would be irrelevant if the instanton were very far from the loop. Likewise, for an n instanton phase matrix with one set $(x_1, \ldots, x_m)$ very far from another set $(x_{m+1}, \ldots, x_n)$, then

$$U^n(x_1 \ldots x_n) \to \Phi\, U^m(x_1 \ldots x_m)\; U^{n-m}(x_{m+1} \ldots x_n) \;, \tag{IV.5}$$

where again we must order the clusters with the lowest values of ϕ to the right. To avoid difficulties later we will define the Φ ordering by placing the matrix with the lowest average value of ϕ to the right. To use the cluster property Eq. (IV.5), we should, of course, prove that the corrections are such that the spatial integrations converge. However, at this time, the solution of n instantons with arbitrary isospin rotations is unknown. Since the isospin integrations certainly influence the correction terms, we can only assume for now that they are small.

Now we define the Euclidean space correlation functions

$$b^n(x_1 \ldots x_n) = \int \prod_{i=1}^{|n|} \frac{d\lambda_i}{\lambda_i^5} \, dR_i \; U^n\left(x_1 \ldots x_{|n|}\right) \exp\left\{K^n(x_i, \lambda_i, R_i; g, \mu)\right\}. \quad \text{(IV.6)}$$

If we define the following matrix amplitudes

$$B_n(V, R) = \prod_{i=1}^{|n|} \int_V d^4x_i \; b^n\left(x_1 \ldots x_{|n|}\right)$$

$$B_o = 2 \quad \text{(IV.7)}$$

$$B(V, R) = \sum_{n=-\infty}^{\infty} \frac{1}{|n|!} B_n(V, R) \; ,$$

where the integrations are over the Euclidean four-volume V, then we have

$$<\chi(c)>^{cl} = \frac{1}{2} \; \mathrm{tr} \; B(V, R) \; . \quad \text{(IV.8)}$$

Now we make the crucial assumption of cluster decomposition. For a set of positions $(x_1, \ldots, x_m)$ very far from another set $(x_{m+1}, \ldots, x_n)$ we assume that

$$b^n(x_1 \ldots x_n) \rightarrow \Phi \, b^m(x_1 \ldots x_m) \, b^{n-m}(x_{m+1} \ldots x_n) \; , \quad \text{(IV.9)}$$

at such a rate so that the spatial integrations converge, i.e., faster than $(x_m - x_{m+1})^{-4}$. Now we can define the cluster decomposed matrix functions

$$
\begin{aligned}
C^1(x) &= b^1(x) \\
C^2(x_1, x_2) &= b^2(x_1, x_2) - \Phi\, C^1(x_1)\, C^1(x_2) \\
C^3(x_1, x_2, x_3) &= b^3(x_1, x_2, x_3) - \Phi \Big[C^1(x_1)\, C^1(x_2)\, C^1(x_3) \\
&\quad + C^1(x_1)\, C^2(x_2, x_3) + C^1(x_2)\, C^2(x_1, x_3) + C^1(x_3)\, C^2(x_1, x_2) \Big]
\end{aligned}
\tag{IV.10}
$$

and in general

$$
b^n(x_1 \ldots x_n) = \Phi \sum \left\{ \prod \left[C^{n_i}(x_1 \ldots) \ldots C^{n_j}(\ldots x_n) \right] \right\}. \tag{IV.11}
$$

The cluster decomposed functions have the property

$$
C^n(x_1 \ldots x_n) \xrightarrow[|x_i - x_j| \to \infty]{} 0 ,
$$

again at a fast enough rate such that the spatial integrations converge. Not if we integrate Eq. (IV.11) over Euclidean space, divide by $|n|!$ and sum n from minus to plus infinity, we obtain

$$
\begin{aligned}
B(V, R) = \Phi \Bigg\{ &\exp \left[\sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^{|n|} \int_V d^4x_i \; C^n(x_1 \ldots x_n) \right] \\
&+ \exp \left[\sum_{n=-1}^{-\infty} \frac{1}{|n|!} \prod_{i=1}^{|n|} \int_V d^4x_i \; C^n(x_1 \ldots x_{|n|}) \right] \Bigg\} .
\end{aligned}
\tag{IV.12}
$$

Since the $C^n(\ _1, \ldots, \ _n)$ are zero if the instantons are either far from each other or far from the quark path, it is easy to see that

$$\prod_{i=1}^{|n|} \int_V d^4x_i\, C^n(x_1 \ldots x_n) = \alpha_n V - \beta_n R + \mathcal{O}(1), \qquad \text{(IV.13)}$$

and therefore, we can easily show that

$$<\chi(c)>^{cl} \underset{R \to \infty}{\sim} \exp\{-a\ R\}\ , \qquad \text{(IV.14)}$$

by bounding Eq. (IV.12) from above and below.

V. THE DENSE ENSEMBLE

In this section, we present a description of a high density ensemble calculation. The technique is the same Monte Carlo computer generation scheme that is used in the theory of liquids.[13] Our major assumption is that the density is sufficiently high so that the short-distance interaction is more important than the long-distance one. The result for the expectation value is the same perimeter law as found in the dilute gas case.

The high density sector is very different from the low density one. First of all, there is a natural cutoff for the size integration. An increasing size is equivalent to decreasing the distance. This can be seen by examining the 'tHooft solution for winding number two. Except for an overall scale factor, as the size of one instanton is increased, the solution looks more and more like a single instanton. The same field configuration can be also accomplished by decreasing the distance between the instantons. But the short-distance repulsion limits the amount of distance that can be decreased. Therefore, the size is limited to how much it can be increased. This is in contrast to the zero density case where although there still is a short-distance repulsion, as the size is increased, the instantons can move away to maintain the same potential. In the non-zero density case, moving instantons must encounter other instantons and be repelled so that there is only a finite range of equipotential of the order of the mean free path divided by the size. This finite mean free path and the short-distance repulsion induces a size limitation.

In addition, if the density is high enough it is a good approximation for liquids and solids to neglect the long range interaction and use only the short-distance one. Therefore, a high density ensemble allows us to make two qualitatively good approximations: limitation of the size and neglect of long range forces. Of course, these crude qualitative arguments should be confirmed by quanitative calculations. However, a complete quantization about a winding number two solution is at present beyond the scope of this paper.

Now let's consider a few simple arguments that clarify the computer calculation. It is easy to see that a rigid uniformly arrayed solid could not even give a perimeter law. Consider an element of path in an instanton solid as shown in Fig. 6. Let this path element's contribution to the phase factor be

$$U_o = e^{i \hat{n} \cdot \vec{\sigma} \theta} , \tag{V.1}$$

then the total phase factor for a large loop made up of p such elements would be given by

$$\chi = \frac{1}{2}\,\mathrm{tr}\;U_o^p = \frac{1}{2}\,\mathrm{tr}\;e^{ip\theta\hat{n}\cdot\vec{\sigma}} = \cos p\theta\;, \tag{V.2}$$

of course, such a rigid solid would only exist for effectively infinite densities.

Now consider an ensemble of separated small size instantons which don't interact, then the phase factor is given approximately by

$$\chi^n \cong \frac{1}{2}\;\mathrm{tr}\;\Phi\left[U^1(x_1)\;U^1(x_2)\;\ldots\;U^n(x_n)\right]\;, \tag{V.3}$$

then the expectation value is given by

$$\begin{aligned}
<\chi(c)>_n^{cl} &\cong \frac{1}{2}\;\mathrm{tr}\;\Phi\prod_{i=1}^{|n|}\;\int_V d^4x_i\;U^1(x_1)\ldots U^1(x_n) \\
&= \frac{1}{2}\;\mathrm{tr}\;\Phi\prod_{i=1}^{|n|}\;\left[\int_V d^4x_i\;U^1(x_i)\right] \\
&= (V-dP)^n \qquad\qquad (\mathrm{V.4}) \\
&= V^n\left(1-\frac{\rho\,dP}{n}\right)^n \\
&\underset{n\to\infty}{\sim} \exp(-\rho\,dP)\;,
\end{aligned}$$

where $\rho = n/V$ is the fixed density. The behavior given by these crude approximations is born out by the computer calculations.

Now we describe in more detail the Monte Carlo generation. First of all, we assume that the density is high enough such that the effective instanton size is limited and that the effective potential can be approximated by only the short distance interaction as discussed in Section II. Remembering that the short distance two-body behavior is given by Eq. (II.4), we approximate the n-body potential by a sum of two-body potentials

$$K^n(x_i, \lambda_i, R_i; g, \mu) \cong \frac{22}{3} \sum_{i<j} \theta\left(\frac{1}{\mu} - |x_i - x_j|\right) \ln |x_i - x_j| \, . \tag{V.5}$$

Of course, using this form can only be justified for ensembles with mean free paths less than the cutoff length $1/\mu$. Because no arbitrarily isospin rotated solution is known, we ignore the isospin integrations and use the multi-instanton 'tHooft solution which is given by

$$A^a_\mu(x) = -\eta^a_{\mu\alpha}\,\partial_\alpha \ln\left[1 + \sum_{i=1}^{|n|} \frac{\lambda_i^2}{(x - x_i)^2}\right] \, . \tag{V.6}$$

According to our previous arguments, we assume that the size integrations lead to a effective size λ.

To eliminate surface effects and better simulate an infinite ensemble with only a small number, periodic boundary conditions are chosen. The ensemble averaging is done as follows. Suppose we start with a configuration $J = \{x_J\}$. Select one instanton at random and consider the configuration L derived from J by moving the selected instanton from its position x_μ to a new position $x_\mu + r_\mu$ where r is a randomly chosen vector in the sphere $|r| < \delta$. If

$$E = \exp\{K^n(x_L) - K^n(x_J)\} \quad , \tag{V.7}$$

is greater than or equal to a random number between zero and one, then the new configuration is L, otherwise it is J. The result is that within statistical errors and feasible computational time limits, the averaged phase factor does not decrease faster than an exponential perimeter.

VI. CONCLUSIONS

We have shown for both dilute and dense configurations of instantons that the expectation value for the phase factor of a large quark loop falls off exponentially with the perimeter. Therefore, unless instantons have non-trivial long range interactions they do not directly contribute to confinement but only to quark mass renormalization. We must conclude then that confinement must come from either more general field configurations or strong coupling.

After the work reported in Section IV was completed, we received a preprint by Callan, Dashen, and Gross.[14] They come to the same conclusions for the dilute gas case. They also present an interesting field configuration which could be relevant to confinement.

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FIGURE CAPTIONS

Fig. 1: A straight line quark path with an instanton a distance b away.

Fig. 2: The phase factor dependence on b for the semicircular path of Fig. 1.

Fig. 3: The phase factor dependence for a circular path as a function of the distance from the center of the loop for an instanton in the plane of the loop and at the same time.

Fig. 4: A circular loop with an instanton on the axis.

Fig. 5: The phase factor dependence for Fig. 4.

Fig. 6: A path element in a hypercubical periodic array of instantons.

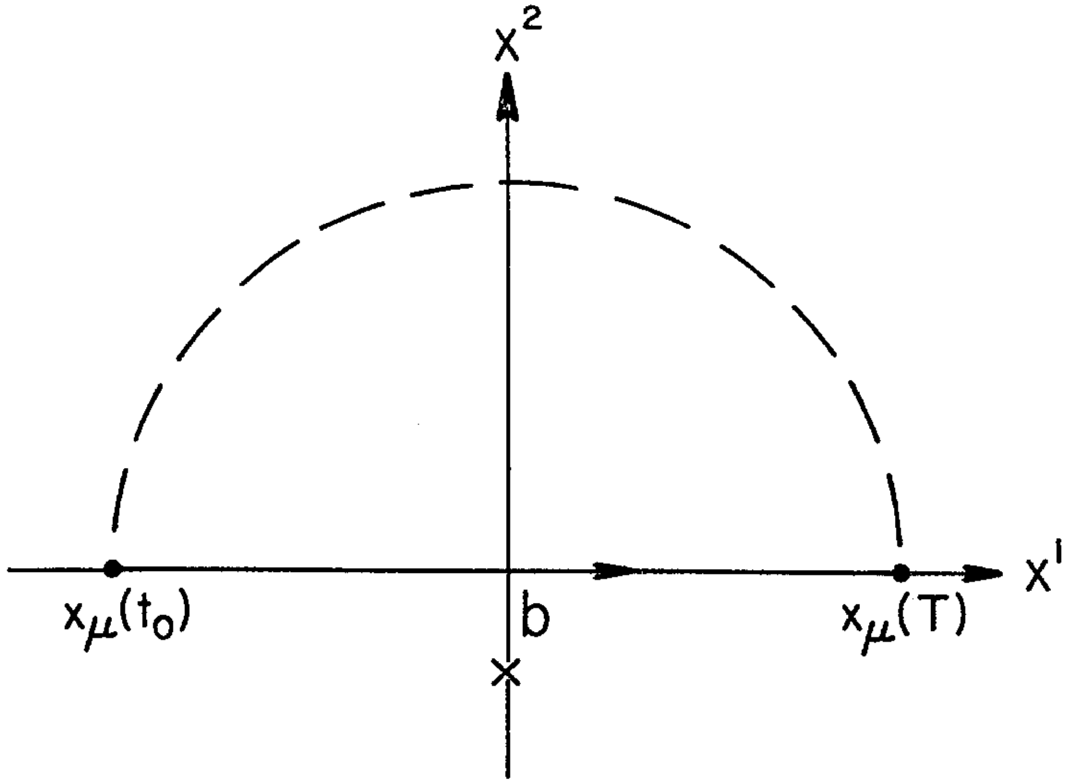

Fig. 1

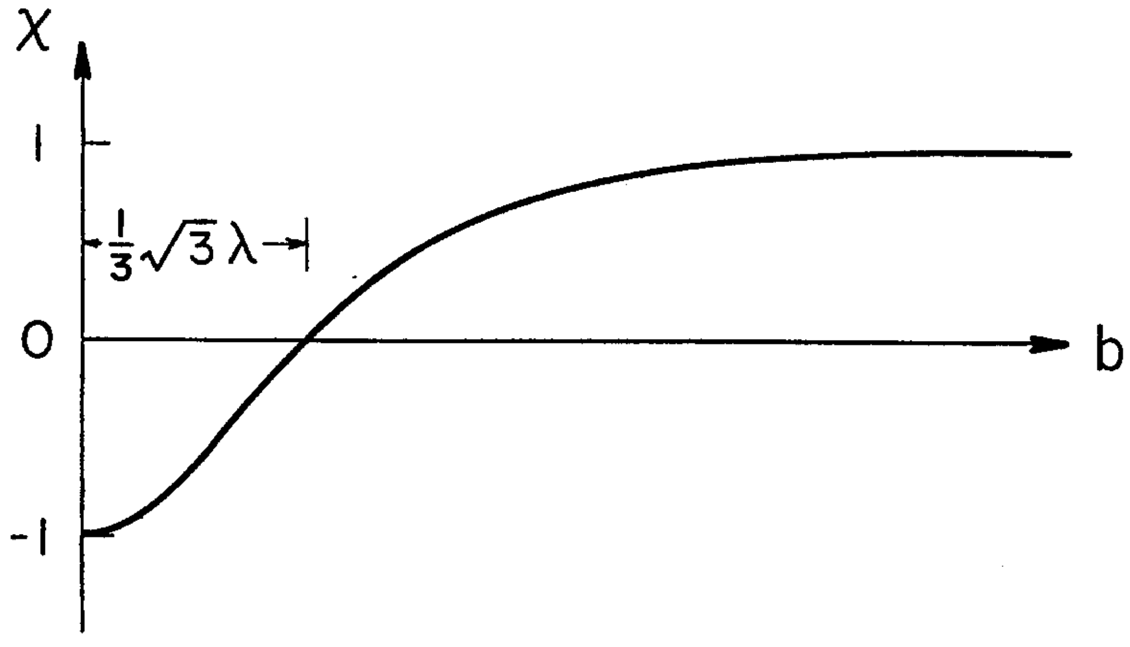

Fig. 2

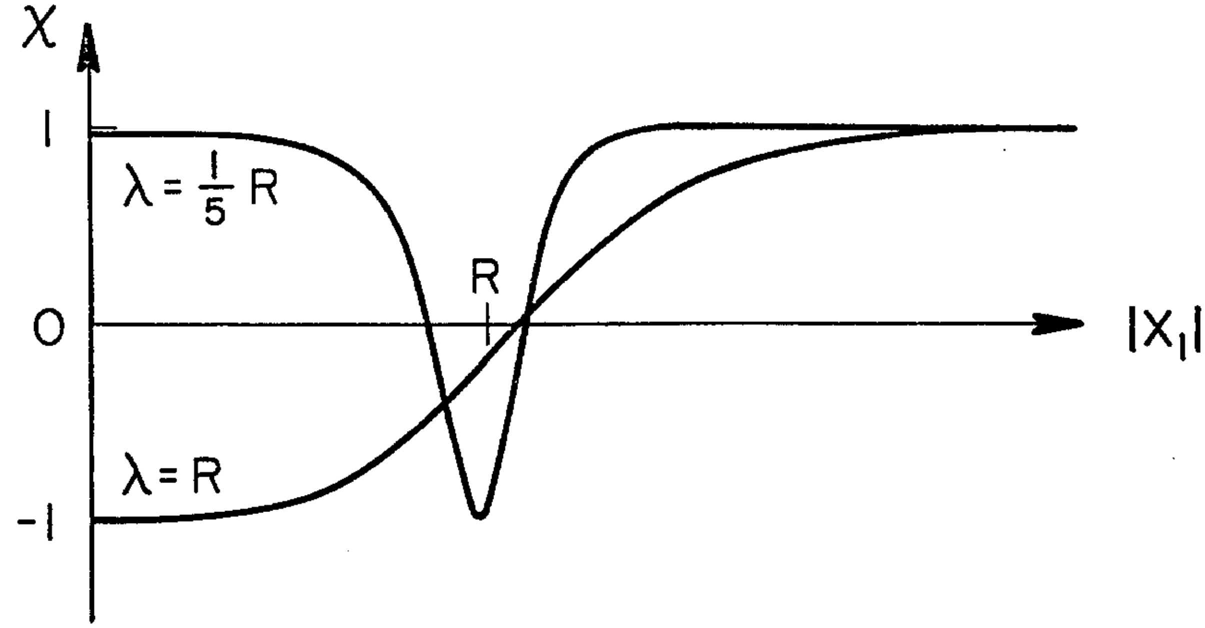

Fig. 3

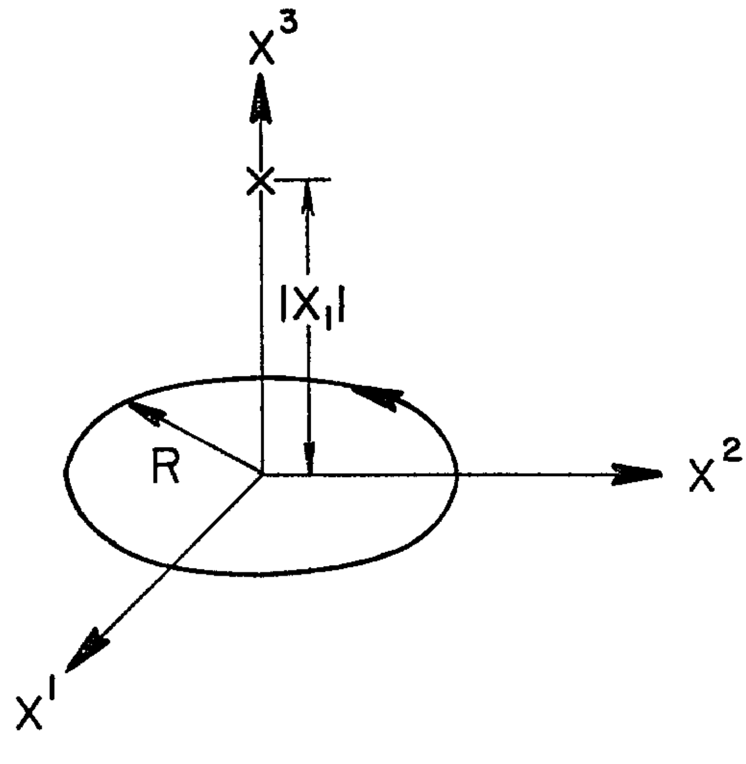

Fig. 4

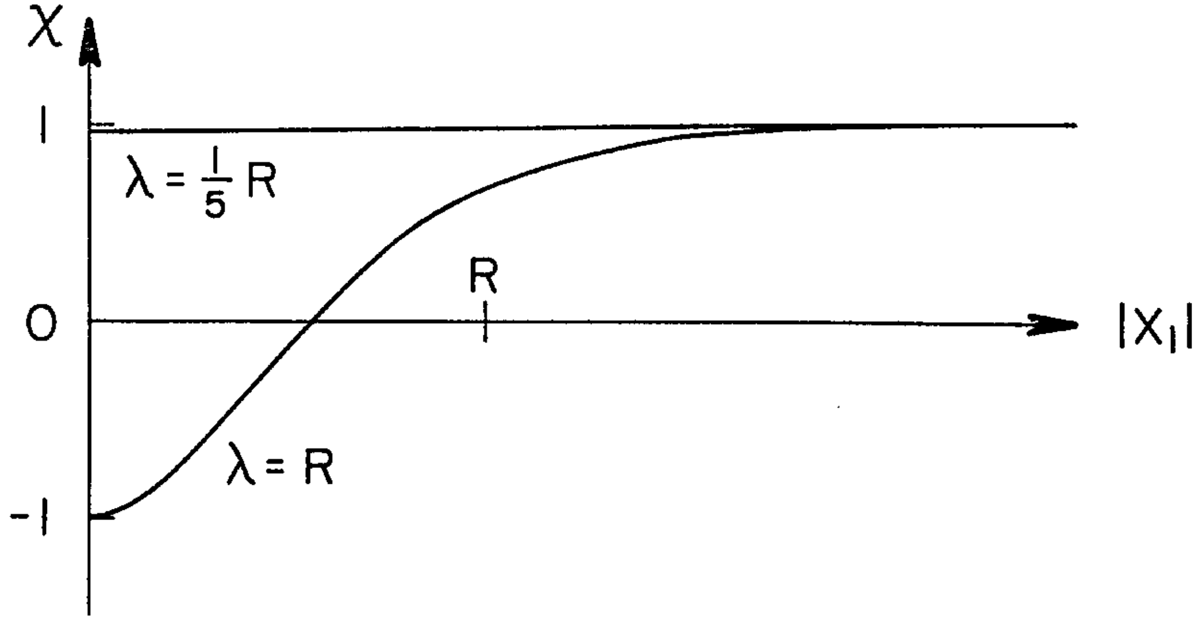

Fig. 5

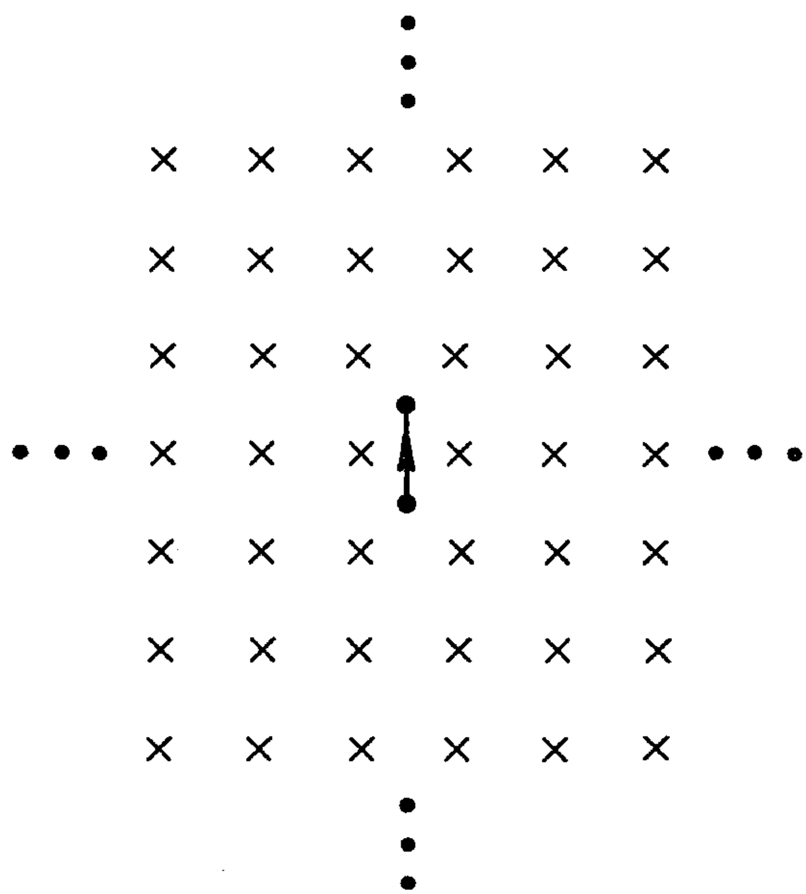

Fig. 6