



## Cancellation of the Zero-Mode Singularities in Soliton Quantization Theory

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### ABSTRACT

We give a proof of an exact cancellation of the zero-mode singularities in each order of the loop expansion determining the quantum corrections to soliton (particle-like) solutions of classical field equations. The cancellation restores an underlying symmetry broken by a specific classical solution, and owe to stability of soliton with respect to small perturbations generated by the symmetry group transformations.

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## I. INTRODUCTION

In the last few years a great deal of attention has been paid to formulation of quantization theory of soliton solutions of nonlinear classical field equations.<sup>1,2</sup> There exists a belief that soliton solution possessing particle-like properties may provide a basis for description of a complex structure of elementary particles.<sup>3</sup> Furthermore, a number of exact topological soliton solutions found recently in the framework of the Yang-Mills equations in Euclidean space can shed some light on the vacuum structure of the quantum theory of gauge fields and on the problem of quark confinement.<sup>4</sup> It was revealed however that calculation of the quantum corrections to the soliton solutions is a rather nontrivial problem. The path integration method as well as perturbation theory failed because of the "zero-mode problem."

To deal with this problem different methods were developed.<sup>1,2,5</sup> Most of them use the description in terms of collective co-ordinates introduced by Bogolubov and Tyablikov in 1949 in the formulation of the quantum theory of polaron<sup>6</sup> and then used in the study of the strong coupling problem and extended objects in quantum field theory.<sup>7</sup>

The zero-mode problem originates from a degeneration of the ground (vacuum) state of a system with respect to transformations of some basic symmetry (translation and/or Lorentz invariance, etc.). This results in appearance of certain zero frequency solutions of the equations determining small variations of a field around the classical soliton solution. Just

as the Goldstone excitations in a theory with a spontaneously broken symmetry those zero-mode solutions can cause an infrared instability and lead to breakdown of the ordinary perturbation theory.<sup>8</sup>

It has been recently shown by Faddeev and Korepin<sup>9</sup> on an example of one and two loop calculations that the zero-mode singularities cancel exactly and the ordinary Feynman rules with an appropriately defined Green function are valid in finding the quantum corrections to the one-soliton solutions. We intend to prove here that the cancellation takes place in each order of the loop expansion for the quantum transition amplitude. An origin of these zero-mode cancellations is closely connected, in our view, with stability of classical soliton solutions with respect to small perturbations generated by transformations of a symmetry group in question.

## II. ONE-DIMENSIONAL QUANTUM MECHANICS PROBLEM

As the arguments of the proof have eventually a rather general character we consider first as the simplest example the non-relativistic quantum mechanical scattering problem given by the action functional

$$A[q(t)] = \int_{-\infty}^{+\infty} dt \left( \frac{1}{2} \dot{q}^2 - v(q) \right) . \quad (1)$$

Then we shall discuss the cancellation of the zero-mode singularities in the perturbation expansion determining the quantum corrections to soliton solutions of field equations.

Assuming the existence of a classical trajectory  $q_0(t)$  satisfying the equation

$$\ddot{q}_0(t) + v'[q_0(t)] = 0 \quad (2)$$

(the dots denote the time derivatives) with an appropriate asymptotic behavior one can develop via the stationary state method the perturbation (quasiclassical) expansion for the transition amplitude in terms of a propagator

$$G = H^{-1} , \quad H = \frac{d^2}{dt^2} + v''[q_0(t)] \quad (3)$$

and vertices

$$V_n = \left( \frac{d}{dq} \right)^n v(q) \Big|_{q = q_0(t)} ; \quad n = 3, 4, \dots \quad (4)$$

As is well known, the formal definition of the propagator (3) faces the zero-mode problem because  $\dot{q}_0 = d/dt q_0(t)$  is an eigenfunction of the operator  $H$  with zero eigenvalue, i. e.  $H\dot{q}_0 = 0$ .

It was pointed out by Faddeev et al.<sup>9</sup> that for non-zero asymptotic velocity  $v = \lim_{t \rightarrow \pm\infty} \dot{q}_0$  a well-defined Green function of the operator  $H$  can be constructed. In this case the function  $\dot{q}_0$  is non-normalizable and corresponds to not a bound but a virtual state of  $H$ .

We shall consider in this note the most singular case of zero asymptotic velocity, i. e. the case of classic trajectories with zero energy

$$E \equiv \frac{1}{2} \dot{q}_0^2 + v(q_0) = 0 \quad . \quad (5)$$

For the class of potentials satisfying the requirement

$$\int_{-\infty}^{+\infty} dq |v(q)|^{\frac{1}{2}} < \infty \quad (6)$$

$\dot{q}_0(t)$  is a localized eigenfunction corresponding to a bound state of  $H$ .

The resolvent  $R = (H + i\epsilon)$  has now a pole at  $\epsilon = 0$ , so the Green function of the problem can be constructed only under the subspace of Hilbert space orthogonal to the vector  $\dot{q}_0 / \|\dot{q}_0\| = \text{const} \cdot \psi_0$ :

$$HG(t, t') = \delta(t - t') - \frac{\dot{q}_0(t)\dot{q}_0(t')}{\|\dot{q}_0\|^2} \quad . \quad (7)$$

It is evident that the Green function  $G$  defined by this equation is non-unique, e.g. the substitution

$$G \rightarrow G + c\psi_0(t)\psi_0(t') \quad (8)$$

gives again an admissible Green function.<sup>9</sup>

This arbitrariness seems to be absent in the functional integration language where the quantum corrections to the transition amplitude in question are given by

$$S_0 = \int [DX(t)] e^{i\Gamma[X(t)]} \quad , \quad (9)$$

where

$$\Gamma[X(t)] = \sum_{n \geq 3} (n!)^{-1} \int V_n X^n(t) dt \quad , \quad (10)$$

and

$$[DX(t)] = \lim_{T \rightarrow \infty} C_T^{-1} e^{i\mathcal{E}[X]} \prod_{-T \leq t \leq T} dX(t); \quad (11)$$

$$C_T = \int e^{i\mathcal{E}[X]} \prod_{-T \leq t \leq T} dX(t); \quad \mathcal{E}[X] = \frac{1}{2} \int XHX dt \quad ,$$

so that  $\int [DX(t)] = 1$ .

As the functional space volume element  $[DX(t)]$  is locally invariant under the substitutions

$$X(t) \rightarrow X(t) + \lambda \psi_0(t) \quad (12)$$

it is obvious that the whole functional integral is not changed by the redefinition of the functional variable  $X(t)$  given by eq. (12).

Now we shall prove that the quasi-classical perturbation expansion generated by

$$S_C = e^{\frac{i}{2} \int (G + C\psi_0 \cdot \psi_0) \frac{\delta}{\delta x \delta x}} \cdot e^{\Gamma[X]} \Big|_{X=0} \quad (13)$$

is independent of an arbitrary constant  $C$ .

Indeed, using the operator identity

$$e^{\frac{C}{2i} \left[ \int \psi_0 \frac{\delta}{\delta x} \right]^2} = \int_{-\infty}^{+\infty} \frac{d\lambda}{\sqrt{4\pi C}} e^{-\frac{\lambda^2}{2C}} \cdot e^{i\lambda \int \psi_0 \frac{\delta}{\delta x}} \quad (14)$$

where the last exponential acts as the functional shifting operator, one gets

$$S_C = \int_{-\infty}^{+\infty} \frac{d\lambda}{\sqrt{-4\pi i C}} e^{-\frac{i\lambda^2}{2C}} \cdot \int [DX(t)] e^{i\Gamma[X + \lambda\psi_0]} \quad (15)$$

Due to the above arguments the functional integral in the eq. does not depend on  $\lambda$  and hence  $S_C$  is independent of  $C$ , i. e.  $S_C = S_0$ . We have only suggested here that the both definitions of the quantum transition amplitude given by eqs. (9) and (13) are equivalent at least at one value of  $C$ , say  $C = 0$ . Thus, the exact transition amplitude, if it exists at

all, is independent of a particular definition of the Green function. One can argue, however, that this proof says nothing about an applicability of perturbation theory because the apparent divergences connected with the zero-mode singularities, the more severe at the higher orders, do not allow to make the shift of the functional variable (12). This seems to invalidate the proof on the perturbation theory level.

We shall show here that the zero-mode singularities as well as the  $\lambda$ -dependent terms cancel exactly in each order of the loop-expansion. The cancellation restores the underlying symmetry of the problem broken by the particular classical solution of dynamic equation. Writing the dynamic variable as a sum of the classical and the quantum parts, i. e.

$$q(t) = q_0(t - a, \dots) + X(t) \quad (16)$$

where we have omitted all but the time mark parameter  $a$ , one can see that the infinitesimal shift of  $a$  and the corresponding transformation of the  $X(t)$  given by eq. (12) lead to an equivalent change of the  $q(t)$  itself.

Owing to this relationship, the following identities are true

$$\int \psi_0 \frac{\delta}{\delta x} e^{i\Gamma[X]} = \not{D}_a e^{i\Gamma[X]}, \quad (17a)$$

$$\left( \int \psi_0 \frac{\delta}{\delta x} \right)^2 e^{i\Gamma[X]} = \left( \not{D}_a^2 + \int \dot{\psi}_0 \left( \frac{\delta}{\delta x} + iHX \right) \right) e^{i\Gamma[X]}, \quad (17b)$$

etc., where

$$\not\partial_a = e^{-i\mathcal{G}[X]} \partial_a e^{i\mathcal{G}[X]} = \partial_a + \frac{i}{2} \int v''' \psi_0 X^2 dt \quad (18)$$

is the time translation generator in the Dirac "interaction" representation.

The following terms of the sequence can be obtained by the induction from the relation

$$T^n = T^{n-1} \cdot \not\partial_a = \not\partial_a \cdot T^{n-1} + (n-1)\dot{T} \cdot T^{n-2} \quad , \quad (19)$$

where

$$T = \int \psi_0 \frac{\delta}{\delta x} \quad \text{and} \quad \dot{T} = -[\not\partial_a, T] \quad . \quad (20)$$

The operator  $\dot{T}$  can be represented in two equivalent forms

$$\dot{T} = e^{-i\mathcal{G}[X]} \int \dot{\psi}_0 \frac{\delta}{\delta x} e^{i\mathcal{G}[X]} = \int \dot{\psi}_0 \left( \frac{\delta}{\delta x} + iHX \right) \quad , \quad (21a)$$

or

$$\dot{T} = \int \dot{\psi}_0 \frac{\delta}{\delta x} + i \int v''' \psi_0^2 X dt \quad , \quad (21b)$$

which both commute with  $T$  since  $\int v''' \psi_0^3 dt = 0$  for a self-conjugated  $H$ .

All the operators on the r. h. s. of those identities allow integration by the parts. Really, it is easy to see that for an arbitrary functional  $F[X]$  after functional averaging according to eqs. (9) and (11) one has

$$(\overline{\not\partial_a F[X]}) = \partial_a (\overline{F[X]}) \quad , \quad (22)$$

and

$$\overline{(\dot{T}F[X])} = \overline{\left( F[X] \delta \left( \int \dot{\psi}_0 / \|\dot{\psi}_0\|^2 X dt - \sigma \right) \right)} \Bigg|_{\substack{\sigma = +\infty \\ \sigma = -\infty}} = 0 . \quad (23)$$

The last result is related to the fact that the  $\dot{\psi}$  is orthogonal in the functional space to the dangerous direction given by the zero-mode eigenfunction  $\psi_0$  and hence

$$[DX(t)] \rightarrow 0, \text{ as } \int \dot{\psi} X dt \rightarrow \pm\infty . \quad (24)$$

It has to be noticed here that the functional averaging of a functional whose dependence on the parameter  $a$  comes through the classical solution  $q_0$ , leads in accordance with the time translation invariance to a result independent of  $a$ .

Consider now the loop expansion given by

$$e^g \frac{1}{2} \Gamma[gX] = \sum_{n=0}^{\infty} g^n L_n[X] . \quad (25)$$

Substituting this into eqs. (17) we obtain after the functional averaging the following net result

$$\overline{T L_n} = \overline{\not{D}_a L_{n-1}} = 0 \quad (26a)$$

$$\overline{T^2 L_n} = \overline{\not{D}_a^2 L_{n-2}} + \overline{\dot{T} L_{n-1}} = 0 \quad (26b)$$

.....

$$\begin{aligned} \overline{T^k_{L_n}} &= \cancel{\delta}_a \overline{T^{k-1}_{L_{n-1}}} + (k-1) \overline{\dot{T} T^{k-2}_{L_{n-1}}} \\ &= \delta_a \left( \overline{T^{k-1}_{L_{n-1}}} \right) + (k-1) \left[ \overline{T^{k-2}_{L_{n-1}}} \cdot \delta \left( \int \dot{\psi}_0 / \|\dot{\psi}_0\|^2 X dt - \sigma \right) \right]_{\sigma=-\infty}^{\sigma=+\infty} = \end{aligned} \quad (26c)$$

The successive terms in this sequence of formulas give variations of the n-loop approximation for the quantum transition amplitude (g) when X(t) undergoes the transformation (12). This completes together with eqs. (22) and (23) the proof of the cancellation of the zero-mode singularities in each order of the loop expansion.

### III. THE ZERO-MODE PROBLEM IN TWO-DIMENSIONAL SCALAR FIELD THEORY

Now we shall briefly discuss the zero-mode problem in the soliton quantization theory. We consider for simplicity the two-dimensional model of a scalar self-interacting field u(x, t) with the action functional

$$A[u] = \int dx dt \left[ \frac{1}{2} \left( u_t^2 - u_x^2 \right) - v(u) \right] . \quad (27)$$

It is assumed that the classical field equation

$$\square u_{cl} + v'(u_{cl}) = 0 \quad (28)$$

admits solutions of the type

$$u_{cl}(x, t) = \phi(r - r_0) ; \quad r = \frac{x - vt}{\sqrt{1 - v^2}} , \quad (29)$$

where  $\phi(r)$  reaches its constant asymptotic values rapidly enough as  $|r| \rightarrow \infty$ , i.e.

$$\phi(r) \rightarrow \phi_{\pm}, \quad r \rightarrow \pm \infty. \quad (30)$$

The quantum corrections to the one-soliton solution can be described via Feynman diagrams in terms of a propagator<sup>1,2</sup>

$$G = H^{-1}; \quad H = \partial_{\tau}^2 + \hat{h}; \quad \tau = \frac{t - vx}{\sqrt{1 - v^2}}, \quad (31)$$

where  $\hat{h} = -\partial_r^2 + v'(u) \Big|_{u = \phi(r - r_0)}$ , and of vertices

$$V_n = \left( \frac{d}{du} \right)^n v(u) \Big|_{u = \phi(r - r_0)}; \quad n \geq 3. \quad (32)$$

Since  $\psi_0 = \partial_{r_0} \phi = -\phi'$  is a localizable eigenfunction of  $h$  with zero eigenvalue and a norm

$$\|\psi_0\|^2 = \int_{-\infty}^{+\infty} dr \psi_0^2 = \int_{\phi_-}^{\phi_+} du (2v(u))^{\frac{1}{2}} < \infty \quad (33)$$

one faces the zero-mode problem in constructing the Green function

$G = H^{-1} = (\partial_{\tau}^2 + \hat{h})^{-1}$ . As was pointed out by Faddeev et al.<sup>9</sup> due to the fact that  $\psi$  is not square integrable function over space-time, the well-defined Green function can be found

$$HG = \delta(x - x')\delta(t - t'); \quad G = G_0 + G_C \quad (34)$$

(hereafter we assume the normalization  $\|\psi_0\|^2 = 1$ ) where

$$G_0 = \frac{1}{2} |\tau - \tau'| \psi_0(r) \psi_0(r') , \quad (35a)$$

$$G_C = \frac{i}{2} \int_{-\infty}^{+\infty} \frac{dk}{\omega_k} e^{-i\omega_k |\tau - \tau'|} \phi_k^*(r') \phi_k(r) ; \quad (35b)$$

are respectively the contributions of the discrete state  $\psi_0$  and of the continuous spectrum of  $\hat{h}$ , i. e.  $\hat{h}\phi_k = \omega_k^2 \phi_k$ ,  $\omega_k = \sqrt{k^2 + \mu^2}$ ,  $\mu$  is a mass of the field  $U(x, t)$ . The usual completeness requirement is assumed here

$$\int_{-\infty}^{+\infty} dk \phi_k^*(r') \phi_k(r) + \psi_0(r') \psi_0(r) = \delta(r' - r) . \quad (36)$$

The only reminiscence of the zero-mode problem is a nonuniqueness of the propagator. So the substitution

$$G \rightarrow G + C \psi_0(r) \psi_0(r') \quad (37)$$

gives again the solution of the eq. (34).

We shall prove here the statement of the paper that this ambiguity (37) does not affect the result in each n-loop approximation.

The quantum corrections to an arbitrary transition matrix element are given by the functional integral

$$S_{fi} = \int [DX(x, t)] e^{i\Gamma[X(x, t)]} \Psi_f^*[X] \Psi_i[X] \quad (38)$$

which is a direct generalization of the corresponding formulas<sup>9-11</sup> in quantum mechanics, with  $X(x, t)$  as the quantum supplement to the one-soliton solution of the field equations and with appropriate initial (final) state functionals  $\Psi_{i(f)}[X]$ . One can choose for example

$$\Psi_{i, f}[X] = \lim_{t \rightarrow \mp \infty} \prod_x \delta(X(x, t) - X_{i, f}(x, t)) \quad , \quad (39)$$

where  $X_{i, f}$  are some eigenfunctions of  $H$ , i. e.

$$X_{i, f}: \psi_{k, \omega}(x, t) = \phi_k(r) \begin{Bmatrix} \cos \omega \tau \\ \sin \omega \tau \end{Bmatrix} \quad . \quad (40)$$

One can see that the expression (38) is invariant under the substitutions  $X \rightarrow X + \delta X$  with the variations  $\delta X$  satisfying the equation  $H \cdot \delta X = 0$  and assuming zero limit when  $t \rightarrow \pm \infty$  to preserve the definition of the initial (final) state functional.

There are at least two such solutions, namely

$$\psi_0(x, t) \equiv \partial_{r_0} \phi = \psi_0(r) \quad , \quad (41a)$$

$$\psi_1(x, t) \equiv \partial_{\theta} \phi = -\tau \psi_0(r) \quad , \quad (41b)$$

where  $\text{ch } \theta = (1 - v^2)^{-\frac{1}{2}}$ . It is worth noting here that the zero asymptotic limit of the both functions  $\psi_{0, 1}(x, t)$  at  $t \rightarrow \pm \infty$  and  $x$  fixed is closely related to the requirement of stability of the soliton solution of classical

field equations<sup>10</sup>. Thus, the general form of variations of  $X$  leaving the functional integral (38) invariant is given by

$$\delta X = \sum_{i=0,1} \lambda_i \psi_i(x, t) \quad (42)$$

with arbitrary constants  $\lambda_i$ .

By a straightforward generalization of the quantum mechanics case one can easily prove now that the results of calculations of the quantum corrections to the one-soliton solution are independent of the ambiguity of the Green function of the general type

$$G \rightarrow G + \sum_{i,j=0,1} C_{ij} \psi_i(x, t) \psi_j(x', t') \quad (43)$$

with arbitrary non-generated matrices  $C_{ij}$ .

Introducing the operators

$$T_i = \int \psi_i \frac{\delta}{\delta X}, \quad i = 0, 1, \quad (44a)$$

$$\not{D}_i = \partial_i + \frac{i}{2} \int v'''' \psi_i X^2, \quad \partial_i = (\partial_{r_0}, \partial_\theta), \quad (44b)$$

$$T_{i/j} = [T_i, \not{D}_j] = - \int \psi_{i,j} \frac{\delta}{\delta X} + i \int v'''' \psi_i \psi_j X, \quad (44c)$$

one can show that the cancellation of the zero-mode singularities in each order of the loop approximation for the transition amplitude is a direct

consequence of the translation ( $r_0 \rightarrow r_0 + a$ ) and the Lorentz ( $\theta \rightarrow \theta + \alpha$ ) invariance owing to the identities

$$(T_i - \not{p}_i) e^{i\Gamma[X]} = 0 \quad , \quad (45a)$$

$$(T_i T_j - \not{p}_i \not{p}_j - T_{i/j}) e^{i\Gamma[X]} = 0 \quad , \quad (45b)$$

etc. In obtaining the following terms of this sequence by an analogy with the quantum mechanics case the commutativity of the operators  $T_i$  and  $T_{j/k}$  has to be taken into account, i. e.

$$[T_i, T_{j/k}] = i \int v''' \psi_i \psi_j \psi_k = 0 \quad . \quad (46)$$

That follows from the requirement  $H$  to be a self-conjugated operator and equation

$$H \psi_{i, j} + v''' \psi_i \psi_j = \partial_j (H \psi_i) = 0 \quad . \quad (47)$$

And finally, we have to emphasize that the stability of the initial (final) state functionals under the variations eq. (42) is essential for the proof.

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