



Planar Bootstrap Without the Dual-Tree Approximation

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ABSTRACT

We consider a dual multiperipheral model at and near $t = 0$, and argue that the usual imposition of a Regge-cluster finite-energy sum rule is probably redundant. Instead we require that the $\pi\pi$ amplitude satisfy the Adler PCAC condition and crossing near $s = t = 0$. This leads to a Reggeon intercept $\alpha(0) \approx 0.5$ for a broad class of models. We then set up a specific Padé approximation to the multiperipheral model. This becomes exact for a factorizable model but takes into account transverse momentum effects and explicitly incorporates the deferred thresholds arising from the production of clusters. We do not make the dual-tree approximation for our Reggeon couplings, which we represent instead by a more general exponential form. If we then assume a linear Reggeon trajectory $\alpha(t)$, self-consistency gives an intercept $\alpha(0) = 0.49$ and a triple-Regge coupling which is in reasonable agreement with experiment. There are no arbitrary parameters in our model.

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I. INTRODUCTION

The first step in any strong-interaction dual-unitarization program, such as the Veneziano $1/N$ expansion,^{1,2} is the "planar bootstrap."³ This is a self-consistent calculation of exchange-degenerate Reggeon poles based on planar-unitarity, and is unaffected by Regge-cut corrections, fixed poles, diffraction and absorption.⁴ One then adds in corrections (cylinder, torus, etc.) which bring in a Pomeron and its interaction with the Reggeon and with itself.^{2,5}

Planar bootstrap calculations are generally carried out within a cluster multiperipheral-model framework. In addition,

(i) Rather simple kinematics is usually assumed. This does not take into account threshold and transverse-momentum effects which are known to be important in certain problems.^{6,7}

(ii) Clusters are related to Regge-exchange using either explicit finite-energy sum rules⁸ or local duality.² We shall argue that such a procedure may sometimes be redundant.

(iii) Unless the planar bootstrap is used merely to fix certain parameters for cylinder and torus calculations, the dual-tree approximation generally has to be made for Regge couplings.^{3,9} It would be desirable to do a calculation in which as many coupling properties as possible are determined by the bootstrap itself.

In the present paper we set up a model in which the above three difficulties do not arise.

In previous planar bootstrap calculations crossing was only applied in a very limited way. In the present calculation we shall apply it directly in the neighborhood of $s = t = 0$. At the same time we require that the $\pi\pi$ amplitude satisfy the Adler PCAC condition. If we then assume a linear Reggeon trajectory and a general exponential form for our Regge couplings we find that we can calculate all the parameters of our model.

In Sec. II we review the dual multiperipheral model. In Sec. III, we discuss the Adler PCAC condition¹⁰ and show that it leads to a Reggeon intercept $\alpha(0) = 0.5$ for a broad class of models; we also discuss crossing near $s = t = 0$. In Sec. IV we discuss a Padé approximation to the multiperipheral model. This involves a box graph, which is evaluated in Sec. V. In Sec. VI we write down our bootstrap results and compare them with the dual-tree approximation. In Sec. VII we include cylinder corrections and calculate the parameters of the Pomeron at intermediate energies. These are then compared with experiment. Finally, in the Appendices we discuss various multiperipheral models in detail, and, in particular, the conditions under which a Padé approximation may be valid.

II. DUAL MULTIPERIPHERAL MODEL

In the multiperipheral cluster-production model, the absorptive part for a two-body process is given by a sum of ladder graphs (Fig. 1). The vertical lines are narrow-resonance clusters a of mass $\sqrt{s_a}$. It was argued in ref. 6 that only a single meson cluster, with $s_a \approx 0.5 \text{ GeV}^2$ and corresponding to the ρ , ω , ϵ , ... peaks, is expected to be important at the sort of intermediate energies where Reggeons play any important role and where duality considerations are expected to apply. In a dual multiperipheral model the horizontal lines are linear combinations of exchange-degenerate pairs of Regge exchanges α .

(A) At the planar bootstrap level we only have uncrossed (planar) quark-duality diagrams of the type shown in Fig. 2. The exchanges then correspond to Regge propagators

$$R = e^{-i\pi\alpha(t)} s^{\alpha(t)} \quad . \quad (2.1)$$

We will assume that $SU(3)$ is exact so that the $\rho - A_2$, $K^* - K^{**}$, $\omega - f$ and $\phi - f'$ pairs are all degenerate, with a linear trajectory

$$\alpha(t) = \alpha_0 + \alpha' t \quad . \quad (2.2)$$

All possible quark diagrams have equal weight.

If we are interested in generating the Pomeron, we must include in the sum of Fig. 1 crossed (cylinder) loops of the type shown in Fig. 3, in addition to the uncrossed loops of Fig. 2. In the former case we then have a Regge propagator

$$R = 1s^{\alpha(t)} \quad . \quad (2.3)$$

(B) In order to obtain an additional constraint on the couplings, the assumption is usually made that the clusters are dual in a finite-energy sum-rule sense to Regge behavior. (See Fig. 4, where the external lines would be either Reggeons or particles). This kind of constraint on sums of ladder graphs was first used a number of years ago within a pion-exchange model⁸ and has recently been applied extensively in the dual multiperipheral approach,^{3,4} often in the extreme local-duality limit.² If Γ_a represents the coupling of the cluster a to the external lines of Fig. 4, we have a relation of the form

$$\Gamma_a = F_a g_1 g_2 \quad (2.4)$$

where F_a is a purely kinematic factor. We shall argue in Appendix A that F_a has approximate factorization properties. This in turn means that Γ_a likewise factorizes.

Explicit expression for F_a can be obtained from finite-energy sum rules (FESR). There are two difficulties which then arise, however.

(i) An explicit FESR involves a separation point between low and high energies which is only known approximately.

(ii) Since Fig. 1 itself has the correct analyticity properties and gives Regge behavior for high s , our multiperipheral model already relates the cluster of Fig. 1(a) to Regge behavior. Thus any further application of an

explicit FESR to the overall amplitude A of Fig. 1 is redundant and may even lead to difficulties. At best it may be a crude and indirect way of imposing crossing, since crossing-symmetric dual amplitudes are known to satisfy simple FESRs. However, a better procedure would be to avoid an explicit FESR altogether in this case and impose crossing directly instead. This will be discussed in the next Section.

III. ADLER ZERO AND CROSSING

Suppose we consider $\pi\pi$ scattering for simplicity, with $m_\pi^2 \approx 0$. Adler's self-consistency condition¹⁰ then requires that the amplitude

$$T(s = 0; t = 0) = 0 \quad . \quad (3.1)$$

Now the planar amplitude satisfies a fixed- t dispersion relation

$$T(s, t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{A(s', t)}{s' - s} \quad , \quad (3.2)$$

where $\nu = \frac{1}{2}(s - u) = s + \frac{1}{2}t$ is the usual crossing-symmetric variable used in finite-energy sum rules. If we expand in ν ,

$$\pi T(s, t) = \sum_{\ell=0}^{\infty} \nu^\ell A_\ell(t) \quad . \quad (3.4)$$

where

$$A_{\ell}(t) = \int_{4m_{\pi}^2}^{\infty} ds' \nu'^{-\ell-1} A(\nu', t) \quad . \quad (3.4)$$

At $t = 0$ this coincides with the usual Mellin transform. We shall see in Sec. IV that it also arises naturally for $t \neq 0$. If we follow Lovelace¹⁰ and apply Eq. (3.1) to Eq. (3.3) we obtain

$$A_0(0) = 0 \quad . \quad (3.5)$$

Further constraints can be obtained by combining Eq. (3.4) with the planar crossing condition

$$T(s, t) = T(t, s) \quad . \quad (3.6)$$

In particular the relation

$$T(0, t) = T(t, 0) \quad , \quad (3.7)$$

which relates the amplitude along the lines $s = 0$ and $t = 0$ in Fig. 5, gives the relations

$$A_0'(0) = \frac{1}{2} A_1(0) , \quad \dots \quad (3.8)$$

We shall now consider two different applications of the above results.

(A) A broad class of simple multiperipheral models gives the form

$$A_{\ell}(t) = n_{\ell}(t) / [1 - K_{\ell}(t)] \quad (3.9)$$

where n_{ℓ} is nonsingular in ℓ , and

$$K_\ell \propto \ln \left[\ell - \alpha_c(t) \right] \quad (3.10)$$

for ℓ in the neighborhood of $\alpha_c(t) = 2\alpha(\frac{1}{4}t) - 1$. Thus, when $\ell \rightarrow \alpha_c$, $K_\ell \rightarrow \infty$ and so $A_\ell \rightarrow 0$. If we identify this zero with that of Eq. (3.5) we see that we must have $\alpha_c(0) = 2\alpha(0) - 1 = 0$, so that $\alpha(0) = 0.5$.

A nonsingular n_ℓ would arise in a completely factorizable model (see Sec. IV). In a more realistic model we might also expect a singularity in n_ℓ at $\ell = \alpha_c$. Since a factorizable model appears to be a good approximation, however (see Appendix A), we would expect such a singularity to be weak, so that $\alpha(0)$ should at least remain close to $\alpha(0) = 0.5$.

(B) From Fig. 1(a.) we have a contribution to the absorptive part

$$W(s, t) = \Gamma(t) \delta(s - s_a) \quad (3.11)$$

Figs. 1(b), (c), ..., on the other hand, give rise to approximate Regge behavior. We can therefore make the usual FESR assumption that

$$A(s, t) \simeq W(s, t) \quad , \quad s < N_0 \quad (3.12)$$

$$\simeq b(t) \nu^{\alpha(t)} \quad , \quad s > N_0 \quad (3.13)$$

where N_0 is midway between s_a and the next cluster above it. If we take both of these to lie on the trajectory (2.2) we then have

$$N_0 \alpha' = 1.5 - \alpha_0 \quad , \quad (3.14)$$

where

$$\alpha' = (1 - \alpha_0)/s_a \quad . \quad (3.15)$$

If we now insert Eqs. (3.12) and (3.13) into Eq. (3.2) and impose the conditions (3.1) and (3.7) we obtain

$$\Gamma(t) = F(t)b(t) \quad , \quad (3.16)$$

where, for small t,

$$F(t) = N_0^{\alpha_0} \frac{s_a}{\alpha_0} \left\{ 1 + \alpha' t \left[\ln N_0 + \frac{\alpha_0}{2\alpha' N_0} + \frac{1}{\alpha' s_a} - \frac{1}{\alpha_0} - \frac{\alpha_0}{2} \frac{(\alpha' N_0)^{-1}}{\alpha_0^{-1}} \right] + \dots \right\} . \quad (3.17)$$

IV. PADÉ APPROXIMATION

Let us consider the usual Froissart-Gribov projection of Eq. (3.2)

$$T_\ell(t) = \frac{1}{2\pi q_t^2} \int_{4m_\pi^2}^{\infty} ds' A(s', t) Q_\ell \left(1 + \frac{s'}{2q_t^2} \right) \quad , \quad (4.1)$$

where $q_t^2 = \frac{1}{4}t - m_\pi^2$ and A is normalized so that

$$A(s, 0) = \lambda^{\frac{1}{2}}(s, m_\pi^2, m_\pi^2) \sigma_{\text{tot}}(s) \quad , \quad (4.2)$$

with $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$. For small t, we can make the asymptotic approximation

$$Q_\ell(z) = B(\ell + 1, \frac{1}{2})(2z)^{-\ell-1} \quad , \quad (4.3)$$

where B is the usual Euler beta function. A combination which is free of kinematic singularities in t is then

$$2\pi B^{-1}(\ell + 1, \frac{1}{2})q_t^{-2\ell} T_\ell(t) \simeq A_\ell(t) \quad , \quad (4.4)$$

where $A_\ell(t)$ is defined as in Eq. (3.4). If we apply this projection to Eq. (3.11), for example, we obtain

$$W_\ell(t) = \Gamma(t)(s_a + \frac{1}{2}t)^{-\ell-1} \quad . \quad (4.5)$$

If we associate a coupling-strength parameter ϕ with each cluster, the sum of Fig. 1 can be written

$$A_\ell(t) = \phi A_\ell^{(1)}(t) + \phi^2 A_\ell^{(2)}(t) + \dots \quad , \quad (4.6)$$

where, for example,

$$\phi A_\ell^{(1)}(t) = W_\ell(t) \quad . \quad (4.7)$$

An $[N, M]$ Padé approximant is then defined in the usual way as the ratio^{12, 13}

$$[N, M] = \frac{\phi n^{(1)} + \dots + \phi^N n^{(N)}}{1 + \phi d^{(1)} + \dots + \phi^M d^{(M)}} \quad (4.8)$$

where $n^{(1)}, \dots, n^{(N)}$ and $d^{(1)}, \dots, d^{(M)}$ are chosen so that an expansion of Eq. (4.8) in powers of ϕ agrees with the expansion (4.6) up to terms of order ϕ^{N+M} . For example,

$$[1, 1] = W_\ell(t)/D_\ell(t) \tag{4.9}$$

where

$$D_\ell(t) = 1 - B_\ell(t)/W_\ell(t) \tag{4.1}$$

and $B_\ell = \phi^2 A_\ell^{(2)}$ is the contribution of Fig. 1(b).

An advantage of the "diagonal" $[N, N]$ Padé approximants is that they satisfy a version of t-channel unitarity in which the particles lying on the exchanged Reggeon trajectories appear in the intermediate state.¹³ If we include π exchange along with the Regge exchanges of Fig. 1, these approximants also satisfy elastic t-channel unitarity exactly in the elastic region. Furthermore, in the case of a factorizable multiperipheral model, it is simple to show that the $[1, 1]$ approximant is in fact exact. In such a model, which we shall justify in Appendix A,

$$\begin{aligned} \phi A^{(1)} &= u_\ell v_\ell \\ \phi^2 A^{(2)} &= u_\ell K_\ell v_\ell \\ \phi^3 A^{(3)} &= u_\ell K_\ell v_\ell u_\ell K_\ell v_\ell, \\ &\dots \end{aligned} \tag{4.1}$$

where u_ℓ and v_ℓ represent external couplings and K_ℓ the internal loops of Fig. 1. (Explicit examples are given in Appendices A and B). The series (4.6) can now be summed exactly to give

$$A_{\ell} = u_{\ell} v_{\ell} / (1 - K_{\ell}) \quad . \quad (4.12)$$

This is exactly what we would obtain from the [1, 1] Padé approximant if we take $A^{(1)}$ and $A^{(2)}$ from Eq. (4.11).

The practical advantage of using the [1, 1] Padé approximant is that we only have to evaluate the first two diagrams of Fig. 1 explicitly. We do not have to first set up a full-fledged multiperipheral integral equation.

V. BOX GRAPH NEAR THE FORWARD DIRECTION

We will assume that the coupling Γ_a in Fig. 6 factorizes so that we can write

$$\Gamma_a = \gamma_{\pi a \alpha}^*(t'', t) \gamma_{\pi a \alpha}(t', t) \quad . \quad (5.1)$$

In the forward direction, Fig. 1(b) then gives

$$B(s, 0) = \frac{1}{16\pi s} \int_{t_-}^{t_+} dt' \left| T_R(t', 0) \right|^2 \theta(s - 4s_a) \quad (5.2)$$

where θ is the usual step function,

$$t_{\pm} = -\frac{1}{4} \left[s^{\frac{1}{2}} \mp (s - 4s_a)^{\frac{1}{2}} \right]^2 \quad (5.3)$$

in the limit of small m_{π}^2 , and

$$T_R(t', t) = \gamma_{\pi a \alpha}^2(t', t) X(t') (\alpha' s)^{\alpha(t')} \quad (5.4)$$

with

$$X(t) = e^{-i\pi\alpha(t)}/\sin \pi\alpha(t) \quad . \quad (5.5)$$

Fig. 1(b) also gives

$$\begin{aligned} \left[\frac{\partial B(s, t)}{\partial t} \right]_{t=0} &= \frac{1}{16\pi s} \int_{t_-}^{t_+} dt' \left\{ \left[\frac{\partial}{\partial t} \left| T_R(t', t) \right|^2 \right]_{t=0} \right. \\ &\quad \left. - \frac{(t' - t_+)(t' - t_-)}{s} \left| \frac{\partial T_R(t', 0)}{\partial t'} \right|^2 \right\} \theta(s - 4s_a) \quad . \quad (5.6) \end{aligned}$$

We will see later how $\gamma_{\pi\alpha\alpha}$ can be related to the $\alpha\alpha\alpha$ triple-Regge coupling. For the present we shall simply parametrize it as

$$\left| X(t') \right|^2 \gamma_{\pi\alpha\alpha}^2(t', t) = G^{1/2} e^{At'} e^{ct/2} \quad (5.7)$$

where G, A and c will all be determined eventually by our bootstrap conditions. Eqs. (5.2) and (5.6) can now be evaluated to give

$$B(s, 0) = \frac{1}{16\pi s} \frac{G}{\tilde{A}} (\alpha' s)^{2\alpha_0} E_0 \quad (5.8)$$

and

$$\left[\frac{\partial B(s, t)}{\partial t} \right]_{t=0} = cB(s, 0) + \frac{1}{16\pi s} GJ(\alpha' s)^{2\alpha_0} E_1 \quad , \quad (5.9)$$

where

$$\tilde{A} = 2 \left[A + \alpha' \ln(\alpha' s) \right] \quad (5.1)$$

$$J = \frac{1}{4} + (\alpha' \pi / \tilde{A})^2 \quad (5.1)$$

$$E_0 = 2e^{\tilde{A}t_1} \sinh \tilde{A}t_2 \quad (5.1)$$

$$E_1 = 4e^{\tilde{A}t_1} \left[\tilde{A}t_2 \cosh \tilde{A}t_2 - \sinh \tilde{A}t_2 \right] / (\tilde{A}s) \quad (5.1)$$

and
$$t_{1,2} = \frac{1}{2}(t_+ \pm t_-) \quad (5.1)$$

To simplify our expressions further we could make the approximations

$$E_0 \approx \left[1 - 4s_a/s \right]^{\frac{1}{2}} \quad (5.1)$$

$$E_1 \approx \left[1 - 4s_a/s \right]^{3/2} \quad (5.1)$$

which are exact for large s and have the correct type of threshold behavior as $s \rightarrow 4s_a$. They are still rather unwieldy, however, and so we will instead make the cruder approximations

$$E_0 \approx \theta(s - s_{L0}) \quad (5.1)$$

$$E_1 \approx \theta(s - s_{L1}) \quad (5.1)$$

where $s_{L0} = 5.333 s_a$ and $s_{L1} = 10.81 s_a$ are the values of s where Eqs. (5.15) and (5.16) attain their maximum values. At the same time we will "exponentiate" \tilde{A} and J

$$\tilde{A} \approx \tilde{A}_{L0} (s/s_{L0})^{2\alpha'/\tilde{A}_{L0}} \quad (5.19)$$

$$J \approx J_{L1} (s/s_{L1})^{-H_{L1}}, \quad (5.20)$$

so that the value and s-derivative is exact at $s = s_{L0}$ and $s = s_{L1}$ respectively. Thus

$$\tilde{A}_{Li} = (\tilde{A})_{s = s_{Li}} \quad (5.21)$$

$$J_{L1} = (J)_{s = s_{L1}} \quad (5.22)$$

and

$$H_{L1} = \frac{4}{\pi} (\alpha' \pi / \tilde{A}_{L1})^3 J_{L1}^{-1} \quad (5.23)$$

If we now make the partial-wave projection (3.4), we obtain

$$B_\ell(0) = \frac{G\alpha'}{16\pi\tilde{A}_{L0}^{2\alpha_0}} \frac{s_{L0}^{2\alpha_0 - 1 - \ell}}{\ell + 1 - 2\alpha_0 + 2\alpha'/\tilde{A}_{L0}} \quad (5.24)$$

and

$$\left[\frac{\partial}{\partial t} \ln B_\ell(t) \right]_{t=0} = c + \left(\ell + 1 - 2\alpha_0 + \frac{2\alpha'}{\tilde{A}_{L0}} \right) \quad (5.25)$$

$$\times \left\{ \tilde{A}_{L0} J_{L1} \frac{(s_{L1}/s_{L0})^{2\alpha_0 - 1 - \ell}}{\ell + 1 - 2\alpha_0 + H_{L1}} - \frac{\frac{1}{2}(\ell + 1)s_{L0}^{-1}}{\ell + 2 - 2\alpha_0 + 2\alpha'/\tilde{A}_{L0}} \right\}$$

VI. BOOTSTRAP RESULTS AND COMPARISON WITH THE DUAL TREE

Eq. (4.9) has an output pole at $\ell = a(t)$ if

$$D_{\alpha(t)}(t) = 0 \quad . \quad (6.1)$$

The corresponding residue is then

$$b(t) = W_{\alpha(t)}(t) \left[\partial D_{\ell}(t) / \partial \ell \right]_{\ell = \alpha(t)}^{-1} \quad . \quad (6.2)$$

We will only consider the value and t-derivative at $t = 0$ and require that this output pole be consistent with the input as given by Eqs. (4.5), (3.16) and (2.2). If $\gamma_{\pi\pi\alpha}^2 = \alpha'^{-\alpha_0}$ we then obtain

$$\alpha_0 = 0.49 \quad , \quad G^{\frac{1}{2}} = 34.6 \gamma_{\pi\pi\alpha}(0), \quad A = 1.99, \quad c = 0.12. \quad (6.3)$$

Our value of α_0 is in good agreement with experiment.

There is no direct unambiguous way of comparing G, A and c with experiment. To relate $\gamma_{\pi\pi\alpha}$ to the $\alpha\alpha\alpha$ triple-Regge coupling g we will use the usual Finite-Mass Sum Rule for the inclusive process $\pi\pi \rightarrow \pi X$

$$\int_0^{\bar{N}_0^2} d\bar{M}^2 \left[\frac{d\sigma}{dt'd\bar{M}^2} - \left(\frac{d\sigma}{dt'd\bar{M}^2} \right)_R \right] = 0 \quad (6.4)$$

with

$$\left(\frac{d\sigma}{dt'd\bar{M}^2} \right)_R = \frac{1}{16\pi s^2} \gamma_{\pi\pi\alpha}^2(t') |X(t')|^2 g(t', t', 0) \gamma_{\pi\pi\alpha}(0) \\ \times (s/\bar{M}^2)^{2\alpha(t')} (\alpha' \bar{M}^2)^{\alpha(0)}, \quad (6.5)$$

where $\overline{N}_0^2 = N_0^2 - m_\pi^2 - t'$, $\overline{M}^2 = M^2 - m_\pi^2 - t'$ and M is the missing mass (see Fig. 7). We will assume that the low M^2 region is dominated by the production of the cluster a so that in the narrow-resonance approximation

$$\frac{d\sigma}{dt' dM^2} = \left(\frac{d\sigma}{dt'} \right)_a \delta(M^2 - s_a) \quad , \quad (6.6)$$

where $(d\sigma/dt')_a$ is the usual differential cross section for $\pi\pi \rightarrow \pi a$, which is given by

$$\left(\frac{d\sigma}{dt'} \right)_a = \frac{1}{16\pi s^2} \gamma_{\pi\pi a}^2(t') |X(t')|^2 \gamma_{\pi a \alpha}^2(t') (\alpha' s)^{2\alpha(t')} \quad . \quad (6.7)$$

From Eqs. (6.4) - (6.7) we obtain

$$g(t', t', 0) = \alpha' \frac{\gamma_{\pi a \alpha}^2(t')}{\gamma_{\pi\pi a}(0)} \frac{\alpha(0) + 1 - 2\alpha(t')}{(\overline{N}_0^2)^{\alpha(0) + 1 - 2\alpha(t')}} \quad . \quad (6.8)$$

This quantity is still difficult to compare with experiment. If we relate it to the fff coupling g_c normalized as in ref. 2 we obtain $g_c \approx 11$ at $t' = 0$. This should be compared with the "experimental" estimate $g_c \approx 6.3$ obtained in ref. 2. However, our calculated value will be lower if we include pion exchange as in ref. 9. Moreover the estimate of ref. 2 relies heavily on a version of two-component duality which may not be valid. In the next section we will consider a less ambiguous confrontation with experiment.

We can also compare our results with the dual-tree approximation, which gives

$$|X(t')|^2_{g(t', t', t)} = N\bar{g}^2 \frac{\Gamma(\alpha(t))\Gamma^2(1 - \alpha(t'))}{\Gamma^2(1 + \alpha(t) - 2\alpha(t'))}, \quad (6.9)$$

where $N = 3$ if we assume that $SU(3)$ is the underlying group. We then obtain $N\bar{g}^2/16\pi = 3.53$, which is again somewhat larger than the usual value. If we use the dual tree to calculate A and c by expanding Eqs. (6.9), (6.8) and (5.7) in t' and t around $t' = 0$ and $t = 0$, we obtain $A = 1.51$ and $c = -1.61$. This should be compared with the bootstrap values obtained in Eq. (6.3). We see that the value of A is relatively close but that c is quite different.

VII. POMERON PARAMETERS AND COMPARISON WITH EXPERIMENT

As discussed in Sec. II, the Pomeron can be calculated by adding in the cylinder loops of Fig. 3. Thus, we must make the replacement

$$B_\ell(t) \rightarrow \hat{B}_\ell(t) = B_\ell(t) + B_\ell^x(t), \quad (7.1)$$

in Eq. (4.10), where B_ℓ^x has the same form as B_ℓ except that we must make the replacement

$$X(t) \rightarrow X^x(t) = 1/\sin \pi\alpha(t) \quad (7.2)$$

in Eq. (5.5). This leads to

$$B_\ell^x(0) = B_\ell(0) \quad . \quad (7.3)$$

On the other hand, Eq. (5.9) must be replaced by

$$\left[\frac{\partial B^x(s, t)}{\partial t} \right]_{t=0} = c B^x(s, 0) + \frac{1}{64\pi s} G(\alpha' s)^{2\alpha_0} E_1 \quad . \quad (7.4)$$

If we use the approximations of Sec. V we then obtain

$$\left[\frac{\partial}{\partial t} \ln B_\ell^x(t) \right]_{t=0} = c + \left(\ell + 1 - 2\alpha_0 + \frac{2\alpha'}{\tilde{A}_{L0}} \right) \left\{ \frac{\tilde{A}_{L0} (s_{L1}/s_{L0})^{2\alpha_0 - 1}}{4 \ell + 1 - 2\alpha_0} - \frac{\frac{1}{2} (\ell + 1) s_{L0}^{-1}}{\ell + 2 - 2\alpha_0 + 2\alpha'/\tilde{A}_{L0}} \right\} \quad . \quad (7.5)$$

(A) From Eqs. (6.1) and (6.2) we now obtain, for small t ,

$$\alpha_{\hat{P}} = 0.75 + 0.97 t \quad (7.6)$$

$$b_{\hat{P}}/b_f = 1.31 (1 - 0.34 t) \quad . \quad (7.7)$$

The f itself becomes extinct in this calculation. Our "Pomeron" \hat{P} , of course, is only an effective pole which describes the cross section for $s \lesssim 50$. To obtain the correct (bare) Pomeron which describes scattering at higher energies, we would have to include higher-mass clusters.^{6, 14} Our intercept is somewhat lower^{15, 5, 9} and our trajectory slope is larger^{2, 6} than in other calculations, but should not be inconsistent with the data for $s \lesssim 50 \text{ GeV}^2$. (See Fig. 4 of ref. 16).

The ratio of Eq. (7.7) is the one for $\pi\pi$ scattering. For pp scattering we must make a correction to take into account the fact that the mass of the end cluster $m_N \neq \sqrt{s_a}$. From ref. 6 or Appendix C we have

$$(b_{\hat{P}}^{\wedge}/b_f^{\wedge})_{pp} = (m_N^2/s_a)^{2(\alpha - \alpha_{\hat{P}}^{\wedge})} (b_{\hat{P}}^{\wedge}/b_f^{\wedge})_{\pi\pi} \quad (7.8)$$

If we assume that the end cluster is the nucleon, we have $m_N \approx 1$ GeV. The resulting $b_{\hat{P}}^{\wedge}$ is similar to the one obtained in refs. 6 and 9 and is consistent with the data within the usual large diffractive-correction uncertainties. If we assume \hat{P} and ω exchange, with ω exchange-degenerate with the f , Eqs. (7.6), (7.7) and (7.8) give an essentially constant total pp cross section in the range $10 \lesssim s \lesssim 50$ GeV².

(B) We can calculate the average cluster multiplicity from the general formula

$$\langle n \rangle_{\text{cluster}} = \phi \left[\frac{1}{b_{\hat{P}}^{\wedge}} \frac{\partial b_{\hat{P}}^{\wedge}}{\partial \phi} + \frac{\partial \alpha_{\hat{P}}^{\wedge}}{\partial \phi} \ln s \right] \quad (7.9)$$

where ϕ is the same parameter as in Sec. IV (see Appendix C). If we use Eqs. (6.1), (6.2) and (7.8), we obtain

$$\langle n \rangle_{\text{cluster}} = 0.85 + 0.39 \ln s \quad (7.10)$$

for pp scattering. The coefficient of $\ln s$ is much smaller than the

usual experimental value. On the other hand it must be remembered that our result is valid only for $s \lesssim 50 \text{ GeV}^2$. For higher s additional clusters are produced and we may have a more rapid overall energy dependence. At $s = 20 \text{ GeV}^2$, Eq. (7.10) gives $\langle n \rangle_{\text{cluster}} \approx 2$. If we assume 2 - 3 particles/cluster, i. e. 1.3 - 2 charged particles/cluster, we obtain a charged-particle multiplicity $\langle n \rangle_{\text{ch}} = 2.7 - 4.0$. Experimentally, $\langle n \rangle_{\text{ch}} \approx 3.2$.

(C) If we use Eq. (7.7) we can use the bootstrap results of Sec. VI to predict the $K^- p \rightarrow K^0 X$ inclusive cross section, which is dominated by the Reggeon-Reggeon-Pomeron graph of Fig. 8. This is given by

$$\frac{d\sigma}{dt'dM^2} = \frac{1}{16\pi s} \gamma_{KK\alpha}^2(t') |X(t')|^2 \frac{1}{\sqrt{6}} g_{\alpha\alpha\hat{P}}^{\wedge}(t't'0) \gamma_{pp\hat{P}}^{\wedge}(0) \times \left(\frac{s}{M^2}\right)^{2\alpha(t')} (\overline{M}^2)^{\alpha_{\hat{P}}(0)}, \quad (7.11)$$

where now $\overline{M}^2 = M^2 - m_p^2 - t'$, and $\gamma_{KK\alpha}^2 = 2\gamma_{KKf}$ with our normalization.

If we again make an end-cluster correction (see ref. 6 or Appendix C)

$$g_{\alpha\alpha\hat{P}}^{\wedge} = \left(\frac{m_N^2}{s_a}\right)^{\alpha(0) - \alpha_{\hat{P}}(0)} \frac{\gamma_{\pi\pi\hat{P}}^{\wedge}}{\gamma_{\pi\pi f}} g_{\alpha\alpha f}, \quad (7.12)$$

which can be calculated using Eq. (7.7), (6.3) and (5.7). The remaining parameters can be extracted from two-body processes. Taking

$\gamma_{\pi\pi f}\gamma_{ppf} \approx 20$ (mb - GeV units) and the quark-duality results

$\gamma_{ppf} = 3/2 \gamma_{\pi\pi f} = 3 \gamma_{KKf}$, we obtain

$$d\sigma/dt'dM^2 = 0.29 \text{ mb/GeV}^4 \quad (7.13)$$

at $t' = 0$. This should be compared with the experimental value

$$d\sigma/dt'dM^2 = 0.33 \text{ mb/GeV}^4 \quad (7.14)$$

at $t' = -0.1 \text{ GeV}^2$.

VIII. CONCLUSION

We have considered a dual multiperipheral model and have imposed the Adler PCAC condition and crossing near $s = t = 0$. We then obtained the following results:

(i) Our Reggeon intercept is $\alpha(0) \approx 0.5$. This is a general consequence of PCAC for a broad class of models, in addition to the specific one used in the present paper.

(ii) The Regge coupling we obtained only has a partial resemblance to the one given by the dual-tree approximation.

(iii) Our Regge couplings and multiplicities are in reasonable accord with experiment.

Further work might involve:

(a) Doing a more accurate calculation which does not make the crude approximations used in Sec. V.

(b) Doing a calculation over a wider range of t , positive as well as negative.

(c) Adding in higher clusters, ^{6,14} including perhaps \bar{p} . The latter would also involve **baryon exchange**.

(d) Alternative or additional crossing conditions could be imposed.

(e) Crossing could be imposed directly on the multiperipheral-model absorptive part, as in Sec. III.

(f) The Padé approach could be applied in an improved way, either by going to higher orders or by summing a different series (see Appendix A).

In the present paper we used a rather simple multiperipheral model with simple resonance clusters. In a more systematic framework,

(A) The multiperipheral cluster-production amplitude could be systematically improved upon, using perhaps Reggeon-calculus techniques.¹⁷

(B) The clusters could be replaced by entire low-energy (low- J) Regge-Regge "scattering" amplitudes, which would themselves be calculated from multiperipheral ladders plus corrections. Energy-thresholds in the Regge exchanges of the multiperipheral chain may have to be put in to avoid double-counting (for low J).

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APPENDIX A. A GENERAL MULTI-REGGE MODEL WITH REGGE-CLUSTER DUALITY

We will give a justification for our use of the [1, 1] Padé approximant (4.9) by considering a general model with Regge-cluster duality. This will also suggest ways of improving our approximation.

We assume that the absorptive part A is given by Fig. 9, where I and II are the initial and final two-body systems in the t channel. At $t = 0$ we then have

$$A_\ell = W_\ell + B_\ell + C_\ell \quad (\text{A.1})$$

where W , B and C correspond to Figs. 9(a), (b) and (c); W is given by Eq. (4.5), whereas

$$B_\ell = \int \rho d\tau' V_\ell^I(\tau') k_\ell(\tau') V_\ell^{II}(\tau') \quad (\text{A.2})$$

and

$$C_\ell = \int \int \rho d\tau_1 \rho d\tau_2 V_\ell^I(\tau_1) k_\ell(\tau_1) \tilde{A}_\ell(\tau_1, \tau_2) k_\ell(\tau_2) V_\ell^{II}(\tau_2) \quad (\text{A.3})$$

where ρ is a number given by phase space and k_ℓ is a two-Regge propagator. In a simple Chew-Goldberger-Low-type model¹⁸ with thresholds, we would have

$$k_\ell(\tau) = \frac{\bar{k}(\tau)}{\ell + 1 - 2\alpha(\tau)} x^{-\ell - 1 + 2\alpha(\tau)} \quad (\text{A.4})$$

where x is a threshold factor.^{7,6}

The "reduced" amplitude \tilde{A} is given by the integral equation of Fig. 10

$$A_\ell(\tau_1, \tau_2) = \tilde{V}_\ell(\tau_1, \tau_2) + \int \rho d\tau' \tilde{V}_\ell(\tau, \tau') k_\ell(\tau') \tilde{A}_\ell(\tau', \tau_2) \quad (\text{A.5})$$

Our cluster coupling functions are given by Eq. (3.11) and

$$V^i(s, \tau) = \Gamma_i(\tau) \delta(s - s_a) \quad , \quad i = I, II \quad , \quad (\text{A.6})$$

$$\tilde{V}(s, \tau_1, \tau_2) = \tilde{\Gamma}(\tau_1, \tau_2) \delta(s - s_a) \quad , \quad (\text{A.7})$$

assuming that all our clusters have the same mass. Cluster-Regge duality, as in Fig. 4 and Eq. (2.4), now gives

$$\Gamma = \gamma_I \gamma_{II} F_W \quad (\text{A.8})$$

$$\Gamma \Gamma_i(\tau) = \gamma_i g(\tau) F_V(\tau) \quad (\text{A.9})$$

$$\tilde{\Gamma}(\tau_1, \tau_2) = g(\tau_1) g(\tau_2) \tilde{F}_V(\tau_1, \tau_2) \quad . \quad (\text{A.10})$$