Fermi National Accelerator Laboratory

FERMILAB-Pub-75/57-THY
July 1975

Quantum Field Theory and the Two-Dimensional Ising Model

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ABSTRACT

We review the relation between the two-dimensional Ising model in the critical domain and the free fermion field theory. The equivalence of the latter with the sine-Gordon model is used to compute corrections away from the critical temperature T_c.

* Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration

I. INTRODUCTION

Ever since Onsager's derivation of the free energy, the Ising model in two dimensions has remained a notorious problem in statistical mechanics. Long ago, Schultz, Mattis and Lieb[2] have shown its equivalence with a free fermion gas in one-dimension. This analogy ca be used to compute its behavior in the critical domain characterized by $T \to T_c$, T_c being the critical temperature, and $\rho >> a$, where a is the lattice spacing, and ρ a typical distance of interest. Furthermore, there is an intimate relation between one-dimensional relativistic massive fermions with or without four fermions coupling, and the sine-Gordon interaction of a Bose field. It turns out that departures from $T = T_c$ can be described by a mass term for the Fermi field. Hence the mass perturbation procedure developed by Coleman[3] gives a means to investigate the leading corrections.

In part II we summarize the Hamiltonian, or transfer matrix, formalism following reference.[2] The third part is devoted to the critical domain. We introduce two non-interacting Ising systems, in order to be able to describe them in terms of a complex Fermi field. The latter enables one to construct a charge current which is related to the gradient of a scalar field. This has the drawback of producing a spurious continuous symmetry at the critical point: the chiral invariance of the charged massless field. Quantities of physical interest are insensitive to this degeneracy.

Mass perturbation is plagued by infrared divergences. A simple example allows us to understand the mechanism of an infrared cutoff. The first non-vanishing correction to the two-point correlation is found to agree with work done using the Toeplitz determinant methods.[4] We hope that further progress along these lines, will enable us to compute higher order corrections. It is gratifying to realize how much the statistical models have in common with relativistic field theories. Each domain can contribute to a better understanding of the other.

II. HAMILTONIAN FORM

A. The Ising Model

At each site of a two-dimensional square lattice (of size L) is attached a dynamical variable $\sigma_i = \pm 1$. The energy of a configuration is a sum of contributions of nearest neighbor pairs:

$$E = -J \sum_{(ij)} \sigma_i \sigma_j \tag{1}$$

The partition function is

$$Z = \sum_{\sigma_i = \pm 1} \exp\left\{-\frac{1}{kT} E(\sigma)\right\} = \sum_{\sigma_i = \pm 1} \exp\left\{\beta \sum_{(ij)} \sigma_i \sigma_j\right\} \tag{2}$$

with $\beta = \frac{J}{kT}$ a natural measure for inverse temperatures. If a large lattice of $N = L^2$ sites approximates the infinite one, we define the free energy per site (up to a factor $-\frac{1}{kT}$):

$$F(\beta) = \lim_{N \to \infty} \frac{1}{N} \ln Z \tag{3}$$

and the correlation functions

$$\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2n}}\rangle = Z^{-1}\sum_{\sigma_i=\pm 1}\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2n}}\exp\left\{\beta\sum_{(ij)}\sigma_i\sigma_j\right\} \tag{4}$$

Onsager[1] was able to compute the function $F(\beta)$, showing in particular that its second derivative is singular at the transition point β_c given by

$$\sinh 2\beta_c = 1 \qquad \beta_c = 0.4407\ldots \tag{5}$$

It is the purpose of the critical theory to investigate the singularity of $F(\beta)$ and the behavior of the correlation functions in the vicinity of $\beta = \beta_c$.

For notational simplicity we shall, most of the time, take the lattice spacing a as a unit of length.

B. Transfer Matrix

The transfer matrix connects the configurations of successive rows of the lattice in such a way that for appropriate boundary conditions:

$$Z = \mathrm{trace}\ V^L \tag{6}$$

The $2^L \times 2^L$ matrix V will be cast into the form $\exp - H$, H being a hamiltonian appropriate for a unit step in an "euclidean time" direction.

The set of row configurations is given a vector space structure in the following way. Orthogonal basis vectors are labelled by the values of L variables σ_k , and denoted

$$|\sigma_1, \sigma_2, \ldots, \sigma_L\rangle$$

This vector space can be identified with the L-th tensor product of a spin ½ , two-dimensional space, with a complete set of operators 1, τ^x, τ^y, τ^z, where the τ's are Pauli matrices and

$$\tau_k^z \, |\sigma_1, \sigma_2, \ldots, \sigma_L\rangle = \sigma_k \, |\sigma_1, \sigma_2, \ldots, \sigma_L\rangle \, . \tag{7}$$

The transfer matrix V is factorized into

$$V = V_1^{1/2} \, V_2 \, V_1^{1/2} \tag{8}$$

In this product, V_1 corresponds to the contribution of those terms in the energy which involve nearest neighbors along the same row:

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_L | V_1 | \sigma_1', \sigma_2', \ldots, \sigma_L' \rangle = \delta_{\sigma_1, \sigma_1'} \delta_{\sigma_2, \sigma_2'} \cdots \delta_{\sigma_L, \sigma_L'} \exp\{\beta \sum_{s=1}^{L} \sigma_s \sigma_{s+1}\}$$

i.e.,

$$V_1 = \exp\{\beta \sum_{s=1}^{L} \tau_s^z \tau_{s+1}^z\} \tag{9}$$

We have adopted the convention that $\tau_{s+L} = \tau_s$, implying periodic boundary conditions. The matrix V_2 takes care of the coupling between rows:

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_L | V_2 | \sigma_1', \sigma_2', \ldots, \sigma_L' \rangle = \prod_{s=1}^{L} \langle \sigma_s | v_2 | \sigma_s' \rangle \, , \tag{10}$$

where v_2 is a two by two matrix given by

$$\langle \sigma | v_2 | \sigma' \rangle = \exp\{\beta \sigma \sigma'\}$$

that is,

$$v_2 = \exp\{\tfrac{1}{2} \ln 2 \sinh 2\beta + \beta^* \tau^x\} \, . \tag{10'}$$

In this last expression β^* is defined by

$$\tanh \beta^* = \exp\{-2\beta\} \, . \tag{11}$$

The fixed point of this transformation, $\beta = \beta^*$, corresponds to the critical value β_c referred to in (5). Thus

$$V_2 = \exp\{\tfrac{L}{2} \ln 2 \sinh 2\beta + \beta^* \sum_{s=1}^{L} \tau_s^x\} \, . \tag{12}$$

We factor out the explicit c-number in (12) which gives a regular contribution to F, and perform a $\frac{\pi}{2}$ rotation around the y-axis in the Pauli-spin ½ space for later convenience. Thus we write:

$$Z = \exp\left\{\frac{N}{2}\ln 2\sinh 2\beta\right\} \text{ trace } W^{L}$$

$$W = W_1^{1/2}\, W_2\, W_1^{1/2}$$

$$W_1 = \exp\left\{\beta \sum_{s=1}^{L} \tau_s^x\, \tau_{s+1}^x\right\} \qquad (13)$$

$$W_2 = \exp\left\{-\beta^* \sum_{s=1}^{L} \tau_s^z\right\}$$

The original variables σ_s correspond now to the operators τ_s^x.

C. Jordan-Wigner Transformation

The Jordan-Wigner transformation is designed to change spin operators into fermion ones as follows. Let

$$\tau^{\pm} = 1/2\,(\tau^x \pm i\,\tau^y)$$

such that

$$\{\tau^+, \tau^-\} = 1, \quad \{\tau^{\pm}, \tau^{\pm}\} = 0 \quad , [\tau^+, \tau^-] = \tau^z .$$

These anticommutation rules are characteristic of Fermi statistics, however $\tau^{\pm}$ operators referring to distinct sites commute.

Fermi operators can however be constructed as

$$c_r = \exp\left\{i\pi \sum_{1}^{r-1} \tau_s^+ \tau_s^-\right\} \tau_r^-$$
$$c_r^+ = \exp\left\{i\pi \sum_{1}^{r-1} \tau_s^+ \tau_s^-\right\} \tau_r^+ , \qquad (14)$$

satisfying

$$\{c_r, c_{r'}\} = \{c^+_r, c^+_{r'}\} = 0 \qquad \{c_r, c^+_{r'}\} = \delta_{r,r'} .$$

The inverse transformation reads

$$\begin{aligned} \tau^-_r &= \exp\{i\pi \sum_1^{r-1} c^+_s c_s\}\, c_r \\ \tau^+_r &= \exp\{i\pi \sum_1^{r-1} c^+_s c_s\}\, c^+_r , \end{aligned} \qquad (14)'$$

from which it follows that:

$$\tau^x_r \tau^x_{r+1} = (c^+_r + c_r)\exp\{i\pi c^+_r c_r\}(c^+_{r+1} + c_{r+1}) = (c^+_r - c_r)(c^+_{r+1} + c_{r+1}) \qquad (15)$$

and

$$\tau^z_r = 2 c^+_r c_r - 1 . \qquad (16)$$

In terms of fermion operators W_1 and W_2 are exponentials of quandratic forms

$$\begin{aligned} W_1 &= \exp\{\beta \sum_1^L (c^+_s - c_s)(c^+_{s+1} + c_{s+1})\} \\ W_2 &= \exp\{\beta^* L - 2\beta^* \sum_1^L c^+_s c_s\} \end{aligned} \qquad (17)$$

A Fourier transformation takes care of translational invariance if one sets:

$$c_r = L^{-1/2} \sum_q e^{iqr} \eta_q \qquad (18)$$

where the momentum q takes discrete values of the form $q = \frac{2\pi}{L} p$, $p = 0, \pm 1, \ldots, \pm \frac{L}{2}$. Since (18) is unitary

$$\{\eta_q, \eta_{q'}\} = \{\eta^+_q, \eta^+_{q'}\} = 0 \qquad \{\eta_q, \eta^+_{q'}\} = \delta_{q,q'} .$$

In terms of these variables

$$\sum_s (c^+_s - c_s)(c^+_{s+1} + c_{s+1}) = \sum_q \left(2\cos q\, \eta^+_q \eta_q + i \sin q\, (\eta^+_q \eta^+_{-q} - \eta_{-q}\eta_q)\right) .$$

This expression was normal-ordered and we used the fact that $\sum_q \cos q = 0$. Similarly $\sum_s c^+_s c_s = \sum_q \eta^+_q \eta_q$.

In the infinite lattice limit the mode at $q = 0$ plays a negligible role for fermions. Thus:

$$
\begin{aligned}
W_1 &= \exp\left\{2\beta \sum_{q>0} \left[\cos q \left(\eta_q^+ \eta_q + \eta_{-q}^+ \eta_{-q}\right) + i \sin q \left(\eta_q^+ \eta_{-q}^+ - \eta_{-q} \eta_q\right)\right]\right\} \\
W_2 &= \exp\left\{\beta^* L - 2\beta^* \sum_{q>0} \left(\eta_q^+ \eta_q + \eta_{-q}^+ \eta_{-q}\right)\right\}
\end{aligned}
\tag{19}
$$

D. Diagonalization

The expressions (19) are suitable for a diagonalization of W. Indeed operators referring to different values of positive q commute, and the problem is reduced to the study of four by four matrices acting on the states

$$|0\rangle \quad , \quad |-q,q\rangle = \eta_{-q}^+ \eta_q^+ |0\rangle \quad , \quad |q\rangle = \eta_q^+ |0\rangle \quad , \quad |-q\rangle = \eta_{-q}^+ |0\rangle ,$$

where $|0\rangle$ is the vacuum for the q and $-q$ modes.

In terms of these states, the relevant operators are represented as

$$
\eta_q^+ \eta_q = \begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \end{pmatrix}
\qquad
\eta_{-q}^+ \eta_{-q} = \begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \end{pmatrix}
$$

$$
\eta_q \eta_{-q} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix}
\qquad
\eta_{-q}^+ \eta_q^+ = \begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix}
\tag{20}
$$

These expressions exhibit the further reduction to a non-trivial problem in the two-dimensional subspace spanned by the vectors $|0\rangle$ and $|-q,q\rangle$. In the $|q\rangle, |-q\rangle,$ subspace the contribution to W is simply the diagonal matrix

$$\exp 2\left(\beta \cos q - \beta^*\right) . \tag{21}$$

In the $|0\rangle, |-q,q\rangle$ subspace we make use of auxilliary Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ (not to be confused with the original σ-variables) with the correspondence

$$\eta_q^+\eta_q\,,\ \eta_{-q}^+\eta_{-q} \longrightarrow \frac{1-\sigma^z}{2}$$

$$\eta_q\eta_{-q} \longrightarrow \frac{\sigma^x+i\sigma^y}{2} \qquad \eta_{-q}^+\eta_q^+ \longrightarrow \frac{\sigma^x-i\sigma^y}{2}\,.$$

The corresponding contribution to W is the two by two matrix w_q equal to

$$w_q = \exp\{2(\beta\cos q-\beta^*)\}\exp\{-\beta\,\sigma.n\}\exp\{2\beta^*\sigma^z\}\exp\{-\beta\sigma.n\}\ , \qquad (22)$$

with the same factor as appeared in (21) and a unit vector n with components $n_x=0,\ n_y=\sin q\ ,\ n_z=\cos q$. With the help of the following definitions (for q > 0) of $\epsilon_q>0$ and ϕ_q

$$\begin{aligned}&\cosh 2\beta\cosh 2\beta^* - \sinh 2\beta\sinh 2\beta^*\cos q = \cosh\epsilon_q\\ &\sinh 2\beta\cosh 2\beta^* - \cosh 2\beta\sinh 2\beta^*\cos q = \sinh\epsilon_q\sin(\phi_q+q)\\ &\sinh 2\beta^*\sin q = \sinh\epsilon_q\cos(\phi_q+q)\end{aligned} \qquad (23)$$

the expression for w_q reads

$$w_q = \exp\{2(\beta\cos q-\beta^*)-\epsilon_q(\sin\phi_q\,\sigma^z+\cos\phi_q\,\sigma^y)\}\ . \qquad (24)$$

In this subspace σ^z stands for $1-\eta_q^+\eta_q-\eta_{-q}^+\eta_{-q}$ and σ^y for $-i(\eta_q\eta_{-q}-\eta_{-q}^+\eta_q^+)$. Both combinations vanish in the $|q\rangle, |-q\rangle$ subspace.

Combining these results for the $\frac{L}{2}$ positive q values we obtain

$$W = \exp\{-\textstyle\sum_{q>0}\epsilon_q\{\sin\phi_q(1-\eta_q^+\eta_q-\eta_{-q}^+\eta_{-q})-i\cos\phi_q(\eta_q\eta_{-q}-\eta_{-q}^+\eta_q^+)\}\} \quad (25)$$

The final step in the diagonalization amounts to a Bogolioubov canonical transformation, to new fermion operators

ξ_q and ξ_q^+ given by

$$\begin{aligned} \eta_q &= e^{-i\pi/4}\left[\cos\left(\frac{\pi}{4}+\frac{\phi_q}{2}\right)\xi_q - \sin\left(\frac{\pi}{4}+\frac{\phi_q}{2}\right)\xi_{-q}^+\right] \\ \eta_{-q} &= e^{-i\pi/4}\left[\cos\left(\frac{\pi}{4}+\frac{\phi_q}{2}\right)\xi_{-q} + \sin\left(\frac{\pi}{4}+\frac{\phi_q}{2}\right)\xi_q^+\right] \end{aligned} \tag{26}$$

leading to

$$W = \exp - H = \exp\left\{-\Sigma_{q>0}\epsilon_q\left(\xi_q^+\xi_q + \xi_{-q}^+\xi_{-q} - 1\right)\right\} , \tag{27}$$

and H describes an assembly of free fermions with "energy" ϵ_q .

E. Free Energy and Correlations

The free energy F involves the computation of trace W^L . Only the largest eigenvalue of W contributes in the limit $L \to \infty$ provided it is non-degenerate, which is the case for $\beta \neq \beta_c$. In turn this means that only the ξ -vacuum state survives in the limit as it is the lowest eigenstate of H . Henceforth brackets will denote averages in this state. Then from (13)

$$F = \frac{1}{2}\ln(2\sinh 2\beta) + \lim_{L\to\infty} L^{-1}\Sigma_{q>0}\epsilon_q .$$

In the infinite L limit the discrete sum is replaced by an integral according to

$$L^{-1}\Sigma_{q>0} \to (2\pi)^{-1}\int_0^\pi dq .$$

Consequently we obtain Onsager's expression in the form

$$F = \frac{1}{2}\ln(2\sinh 2\beta) + \frac{1}{2\pi}\int_0^\pi dq\,\epsilon_q , \tag{28}$$

with ϵ_q given implicitly by (23). As long as $\beta \neq \beta_c$, then $\beta \neq \beta^*$ and $\epsilon_q \geq 2|\beta - \beta^*|$, which means that the vacuum is an isolated point in the spectrum. As $\beta \to \beta_c$ the energy gap goes to zero and the infrared region becomes predominant in the calculation of physical quantities.

We shall concentrate our attention here to the two-point correlation at "equal time", meaning along the same row. As we expect an isotropic behavior close to the critical point, this is not a serious limitation in this case.

Let $r < r'$ refer to points on the same row; the thermal average $\langle \sigma_r \sigma_{r'} \rangle$ is given in terms of a ($\dot{\xi}$) vacuum expectation value as

$$\langle \sigma_r \sigma_{r'} \rangle = \langle (c_r^+ + c_r) \exp\left\{ i\pi \sum_{r}^{r'-1} c_s^+ c_s \right\} (c_{r'}^+ + c_{r'}) \rangle$$

$$= \langle (c_r^+ - c_r) \exp\left\{ i\pi \sum_{r+1}^{r'-1} c_s^+ c_s \right\} (c_{r'}^+ + c_{r'}) \rangle \;.$$

Due to the anti-commutativity of the c's the right hand side operators are hermitian. Even though the underlying dynamics has been brought to the simple form (27) which means that the vacuum has a simple structure, the computation of correlations seem to involve complicated expressions in terms of fermion degrees of freedom.

III. CRITICAL REGION

A. Expansion Near $\beta = \beta_c$

Close to $\beta = \beta_c$, singularities appear involving long range correlations. The discrete nature of the lattice is washed out in the large distance behavior of correlation functions, and continuous euclidean invariance is restored in this limit. We set

$$\beta = \beta_c - \frac{m}{4} \quad , \tag{30}$$

where in units of the lattice spacing $m \ll 1$, and from (11) to leading order

$$\beta^* = \beta_c + \frac{m}{4} \quad . \tag{30'}$$

Here $m > 0$ $(m < 0)$ corresponds to $T > T_c$ $(T < T_c)$. Only long wavelength with respect to the lattice spacing, i.e., small q, are of interest. Thus it is legitimate to expand ϵ_q near $q = 0$. From (23)

$$\epsilon_q^2 = m^2 + q^2 \ . \tag{31}$$

This is the relativistic dispersion relation with the identification of $|m|$ with the fermion mass. From the relations (23) we also find in this approximation

$$\sin\phi_q = -\frac{m}{\sqrt{m^2+q^2}} \qquad \cos\phi_q = \frac{q}{\sqrt{m^2+q^2}} \quad , \tag{31'}$$

which allow us to write W in the form

$$W = \exp - H = \exp\Big\{ -\sum_{q>0} \big[m\left(\eta_q^+ \eta_q + \eta_{-q}^+ \eta_{-q} - 1\right) + iq\left(\eta_{-q}^+ \eta_q^+ - \eta_q \eta_{-q}\right)\big]\Big\} \tag{32}$$

We still keep discrete values for q, recalling that in the large L limit $\sum_q \to (L/2\pi)\int_{-\pi}^{+\pi} dq$ and the transition from discrete (η, η^+) to continuous $(\tilde{\eta}, \tilde{\eta}^+)$ fermion operators amounts to:

$$\tilde{\eta}_q = \sqrt{\frac{L}{2\pi}}\, \eta_q \qquad \tilde{\eta}_q^{\,+} = \sqrt{\frac{L}{2\pi}}\, \eta_q^+ \quad ; \tag{33}$$

the continuous operators fulfill anticommutation relations with δ-function replacing Kronecker δ's.

A quantity that can immediately be computed is the singularity of the specific heat, i.e., the second derivative of F . We call ΔF the departure of F from its value at $\beta=\beta_c$, excluding the regular $\frac{1}{2}\ln 2\sinh 2\beta$ piece. Hence

$$\Delta F(m)=\frac{1}{2\pi}\int_0^{\pi}dq\left(\sqrt{q^2+m^2}-q\right)=-\frac{m^2}{4\pi}\ln|m|+O(m^2)\,, \tag{34}$$

which shows the logarithmic singularity of the specific heat.

At $\beta=\beta_c$ the mass m is equal to zero, which implies $\phi_q=0$. Using (26) the diagonalization of the Hamiltonian is performed through

$$\eta_q=\frac{e^{-i\pi/4}}{\sqrt{2}}\left(\xi_q-\xi_{-q}^{+}\right)\qquad \eta_{-q}=\frac{e^{-i\pi/4}}{\sqrt{2}}\left(\xi_{-q}+\xi_q^{+}\right) \tag{35}$$

The original fermion operators c_r can then be written

$$c_r=\frac{e^{-i\pi/4}}{\sqrt{2}}\left(\psi_1+i\psi_2\right) \tag{36}$$

with

$$\begin{aligned}\psi_1(r)&=\frac{1}{\sqrt{L}}\sum_{q>0}\left(e^{iqr}\xi_q+e^{-iqr}\xi_q^{+}\right)\rightarrow\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{+\pi}dq\left(e^{iqr}\tilde{\xi}_q+e^{-iqr}\tilde{\xi}_q^{+}\right)\theta(q)\\ i\psi_2(r)&=\frac{1}{\sqrt{L}}\sum_{q<0}\left(e^{iqr}\xi_q-e^{-iqr}\xi_q^{+}\right)\rightarrow\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{+\pi}dq\left(e^{iqr}\tilde{\xi}_q-e^{-iqr}\tilde{\xi}_q^{+}\right)\theta(-q)\,.\end{aligned} \tag{37}$$

In dimensional units the limits of integration are $\pm\pi/a$, and can be replaced by $\pm\infty$ when no ultraviolet divergences are encountered. The hermitean fields ψ_1 and ψ_2 are the two components of a relativistic Majorana spinor field,[5] if we allow for Minkowskian time development

$$\psi(r,t)=e^{iHt}\psi(r)e^{-iHt}, \tag{38}$$

which leads to

$$\psi_1(r,t)=\psi_1(r-t)\qquad \psi_2(r,t)=\psi_2(r+t)\,. \tag{39}$$

If we define the following γ matrices

$$\gamma^0 = \gamma^{0\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \gamma^{1\dagger} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^5 = \gamma^{5\dagger} = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad g^{00} = -g^{11} = 1 \quad , g^{01} = g^{10} = 0 \tag{40}$$

the field ψ satisfies the Dirac equation

$$i\gamma . \partial \psi = 0 \tag{41}$$

In the discrete version ψ satisfies ordinary anticommutation rules

$$\{\psi_\alpha(r), \psi_\beta(r')\} = \delta_{\alpha\beta}\, \delta_{rr'} \;,$$

and its equal time Wightman functions are

$$\langle\psi_1(r)\,\psi_1(r')\rangle = \langle\psi_2(r)\psi_2(r')\rangle^* = \frac{i}{2\pi}\;\frac{1-e^{i\pi(r-r')}}{r-r'} \rightarrow \frac{i}{2\pi(r-r')} \tag{42}$$

$$\langle\psi_1(r)\,\psi_2(r')\rangle = 0 .$$

As long as the spacing is kept finite we find from (42) that $\langle\psi_1^2(r)\rangle = \langle\psi_2^2(r)\rangle = \frac{1}{2}$. The limiting form $i[2\pi(r-r')]^{-1}$ is only valid for large separations, and is then identical with the result obtained using continuous field theory.

B. Two-Point Correlation Function at the Critical Point

To express the two-point correlation function in terms of ψ we observe that since

$$e^{i\pi c^\dagger c} = 1 - 2c^\dagger c = (c^\dagger + c)(c^\dagger - c)$$

one can rewrite (29) as

$$\langle\sigma_r\sigma_{r'}\rangle = \langle [(c_r^+ - c_r)(c_{r+1}^+ + c_{r+1})][(c_{r+1}^+ - c_{r+1})(c_{r+2}^+ + c_{r+2})]\cdots$$
$$\cdots [(c_{r'-1}^+ - c_{r'-1})(c_{r'}^+ + c_{r'})]\rangle \tag{43}$$
$$= \langle \psi_r^{(-)}\psi_{r+1}^{(+)}\psi_{r+1}^{(-)}\psi_{r+2}^{(+)}\cdots\psi_{r'-1}^{(-)}\psi_{r'}^{(+)}\rangle$$

where from (36)

$$c^+ + c = \psi_1 + \psi_2 = \psi^{(+)} \qquad c^+ - c = i(\psi_1 - \psi_2) = \psi^{(-)} \tag{44}$$

In the continuous limit

$$\langle \psi_r^{(-)}\psi_{r'}^{(+)}\rangle = \langle \psi_r^{(+)}\psi_{r'}^{(-)}\rangle$$

This means that if we use another set of fields φ isomorphic to ψ we also have:

$$\langle\sigma_r\sigma_{r'}\rangle = \langle \varphi_r^{(+)}\varphi_{r+1}^{(-)}\varphi_{r+1}^{(+)}\varphi_{r+2}^{(-)}\cdots\varphi_{r'-1}^{(+)}\varphi_{r'}^{(-)}\rangle \tag{45}$$

The reason for introducing a second set of Majorana fields is the following. The aim is to construct a complex Dirac massless field which allows the existence of non-vanishing currents. These currents are the appropriate gradients of scalar massless fields. Calculations with the latter are greatly simplified. This trick is related but not equivalent to the method used in Ref. 6 and 7.

We let φ and ψ anticommute and introduce the common vacuum for both fields. The square of the correlation function reads:

$$\langle\sigma_r\sigma_{r'}\rangle^2 = \langle \psi_r^{(-)}\psi_{r+1}^{(+)}\cdots\psi_{r'-1}^{(-)}\psi_{r'}^{(+)}\rangle\langle \varphi_r^{(+)}\varphi_{r+1}^{(-)}\cdots\varphi_{r'-1}^{(+)}\varphi_{r'}^{(-)}\rangle \tag{46}$$
$$= (-1)^\rho \langle (\psi_r^{(-)}\varphi_r^{(+)})(\psi_{r+1}^{(+)}\varphi_{r+1}^{(-)})(\psi_{r+1}^{(-)}\varphi_{r+1}^{(+)})\cdots(\psi_{r'}^{(+)}\varphi_{r'}^{(-)})\rangle$$

where $\rho = r' - r$. If J and K denote the combinations

$$J = \psi^{(+)}\varphi^{(-)} \qquad K = \psi^{(-)}\varphi^{(+)} \tag{47}$$

they satisfy

$$J = J^{+} \qquad K = K^{+} \qquad J^2 = K^2 = 1 \qquad [J,K] = 0\ ,$$

hence

$$J = e^{i\frac{\pi}{2}(J-1)} \qquad K = e^{i\frac{\pi}{2}(K-1)}$$

This allows one to write

$$\langle \sigma_r \sigma_{r'} \rangle^2 = \langle \exp\left\{ i\frac{\pi}{2}\left[K_r + \sum_{r+1}^{r'-1} (J_s + K_s) + J_{r'} \right]\right\} \rangle$$

We note a boundary effect: the first K and the last J are not paired. However this can be neglected in the limit $\rho \to \infty$ as a careful study shows using the determinantal expression for the correlation. For our purposes we can write

$$\langle \sigma_0 \sigma_\rho \rangle^2 = \langle \exp\left\{ i\frac{\pi}{2} \sum_{1}^{\rho} (J_s + K_s) \right\} \rangle \tag{48}$$

With D standing for the complex Dirac massless field

$$D = \frac{\psi + i\varphi}{\sqrt{2}} \quad , \quad \{ D_\alpha(r), D^{+}_\beta(r') \} = \delta_{\alpha\beta}\, \delta_{rr'} \tag{49}$$

we recognize that $\frac{J+K}{2}$ is the space component of the charge current j :

$$\frac{1}{2}(J+K) = \frac{1}{2}\left(\psi^{(-)}\varphi^{(+)} + \psi^{(+)}\varphi^{(-)} \right) = i(\psi_1\varphi_1 - \psi_2\varphi_2)$$
$$= D^{+}_1 D_1 - D^{+}_2 D_2 = \bar{D}\, \gamma^1 D = j^1 . \tag{50}$$

From Klaiber's work[8] it is known that in the continuum limit the current j^μ can be expressed as the gradient of a free massless field Φ with a suitable infrared regularization. To conform to standard normalization we write

$$j^\mu = \frac{1}{\sqrt{\pi}}\, \partial^\mu \Phi \tag{51}$$

The infrared regularization is irrelevant for $\partial^\mu \Phi$. We need the equal time two-point function of Φ. This is related to the one for ψ as follows

$$\langle j'(r)\, j'(r')\rangle = \langle \psi_1(r)\psi_1(r')\rangle^2 + \langle \psi_2(r)\psi_2(r')\rangle^2$$
$$= \frac{-1}{2\pi^2}\frac{1}{(r-r')^2} = \frac{-1}{2\pi^2}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}\ln|r-r'| \,. \tag{52}$$

Inside the logarithm $|r-r'|$ could be multiplied by any mass μ without affecting the result. Thus we set

$$\langle \phi(r)\,\phi(r')\rangle = \frac{-1}{2\pi}\ln \mu|r'-r| \,. \tag{53}$$

This is quite a natural definition. Indeed consider a massive sclar field in two-dimensional space-time. In the euclidean region if μ' stands for the mass and if r and r' are two-dimensional vectors the Wightman function can be written:

$$\int \frac{d^2k}{(2\pi)^2}\,\frac{e^{ik(r-r')}}{k^2+\mu'^2} = \frac{1}{2\pi}K_0(\mu'|r-r'|)\,,$$

with K_0 the modified Bessel function given for small x by

$$K_0(x) = -\left(\ln\frac{x}{2}+\gamma\right) + O(x^2\ln x)\,.$$

If we assume that $\mu'\to 0$ and if we set $\mu = \frac{1}{2}\mu' e^{\gamma}$, the uniform approximation for $|r-r'| \ll \frac{1}{\mu}$ is indeed given by (53). Approximating sums by integrals as $L\to\infty$, and using (51) we see that

$$\langle\sigma_0\sigma_\rho\rangle^2 = \langle \exp\left\{ i\sqrt{\pi}\left(\Phi(0) - \Phi(\rho)\right)\right\}\rangle \tag{54}$$

The exponentials commute for $\rho\neq 0$ but they require a multiplicative renormalization.[3] This means that we do not try to compute the absolute normalization of the correlation function. Thus we replace $e^{i\beta\Phi(r)}$ by

$$\mathcal{N}\, e^{i\beta\Phi(r)} = \mu^{\frac{\beta^2}{4\pi}} :e^{i\beta\Phi(r)}: \,. \tag{55}$$

The : : mean Wick ordering with respect to the mass $\mu' = 2\mu e^{-\gamma}$ of a massive scalar field in the limit $\mu\to 0$ which amounts to use (53) as the two-point function. Our definition, slightly different from Coleman's,[3] implies that $\mathcal{N} e^{i\beta\Phi}$ is given the effective dimension

$$\dim\left(\mathcal{N} e^{i\beta\Phi}\right) = \beta^2/4\pi \tag{56}$$

Further calculations are always understood in the limit $\mu \to 0$.
In this way Wick's theorem yields

$$\langle \mathcal{N} e^{i\beta_1\Phi(r_1)} \cdots \mathcal{N} e^{i\beta_n\Phi(r_n)} \rangle = \mu^{\frac{\Sigma\beta_i^2}{4\pi}} \prod_{j<k} \left(\mu|r_j - r_k|\right)^{\frac{\beta_i\beta_j}{2\pi}} \tag{57}$$

This contains a factor $\mu^{\frac{(\Sigma\beta_i)^2}{4\pi}}$, and hence vanishes if $\Sigma\beta_i \neq 0$.

It follows at once that, up to an unknown constant factor, set equal to one for convenience

$$\begin{aligned} \langle \sigma_0 \sigma_\rho \rangle^2 &= \langle \mathcal{N} e^{i\sqrt{\pi}\Phi(0)} \mathcal{N} e^{-i\sqrt{\pi}\Phi(\rho)} \rangle \\ &= \rho^{-1/2} \end{aligned} \tag{58}$$

It is known in general that the correlation function is positive (Griffith's first inequality); consequently one can take the square root of (58) and one finds:

$$\langle \sigma_0 \sigma_\rho \rangle = \rho^{-1/4} \tag{59}$$

C. Mass Perturbation

Below T_c $(m<0)$, the Ising system develops a spontaneous magnetization. This means that for $m<0$ fixed, and $\rho \to \infty$, the correlation function tends to a finite limit, equal to the square of the magnetization. We shall however investigate a different limit such that $\rho \to \infty$, $m \to 0$ but $m\rho$ remains finite.

Under such circumstances the spontaneous magnetization vanishes and:

$$\langle \sigma_0 \sigma_\rho \rangle = \rho^{-1/4} F_\pm(|m|\rho) \tag{60}$$

where the + or - sign refer respectively to $m>0$ or $m<0$, and $F_\pm(0) = 1$.

We shall only attempt to compute the leading correction to $F_\pm(t)$ for t small, using mass perturbation.

To illustrate the care required to deal with such a perturbation and its infrared singularities, let us first treat a simple example. Assume for the moment $m>0$ and express the Hamiltonian given in (32)

$$H = \Sigma_{q>0}\left[iq(\eta^+_{-q}\eta^+_q - \eta_q\eta_{-q}) + m(\eta^+_q\eta_q + \eta^+_{-q}\eta_{-q})\right] = H_0 + H_m ,$$

in terms of the Majorana field ψ ($\bar{\psi}$ stands for $\psi\gamma^0$):

$$\begin{aligned} H_0 &= \frac{-i}{2}\int dr\, \bar{\psi}\gamma^1\partial_1\psi \\ H_m &= \frac{-im}{2}\int dr\, \bar{\psi}\gamma^5\psi \ . \end{aligned} \tag{61}$$

The mass term occurs with an unconventional $-i\gamma^5$ matrix but this does not play any significant role here. Having introduced two uncoupled fields ψ and φ , the total Hamiltonian is:

$$H(\psi) + H(\varphi) = \int dr\, \mathcal{H}(r) \tag{62}$$

The Hamiltonian density $\mathcal{H}$ expressed in terms of the complex Dirac field D, reads:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_m \qquad \mathcal{H}_0 = -i\bar{D}\gamma^1\partial_1 D \qquad \mathcal{H}_m = -im\bar{D}\gamma^5 D \ . \tag{63}$$

To derive (61) and (63) we have used a Fourier analysis of D adapted to the mass zero case

$$D_1(r) = \frac{1}{\sqrt{2\pi}} \int dq \left(e^{iqr} a_q + e^{-iqr} b_q^+ \right) \theta(q)$$
$$iD_2(r) = \frac{1}{\sqrt{2\pi}} \int dq \left(e^{iqr} a_q - e^{-iqr} b_q^+ \right) \theta(-q). \tag{64}$$

If ξ'_q stands for the operators pertaining to the field φ and $\tilde{\xi}'_q$ are their continuous version, then of course

$$a_q = \frac{1}{\sqrt{2}} \left(\tilde{\xi}_q + i \tilde{\xi}'_q \right) \qquad b_q^+ = \frac{1}{\sqrt{2}} \left(\tilde{\xi}_q^+ + i \tilde{\xi}'^+_q \right)$$

satisfy the ordinary anticommutation rules. The choice of (64) as a Fourier analysis seems inappropriate for the massive theory. Indeed a canonical Bogslioubov transformation can be performed from a's and b's to A's and B's adapted to massive fermions. From (63) the Minkowskian time evolution of D is governed by the Dirac equation

$$-i \frac{\partial}{\partial t} D(r,t) = \int dr' \left[\mathcal{H}(r',t), D(r,t) \right]$$
$$= i \left(\gamma^5 \frac{\partial}{\partial r} D(r,t) + m \gamma^1 D(r,t) \right)$$

Explicitly

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right) D_1 = m D_2$$
$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r} \right) D_2 = -m D_1 \tag{65}$$

implying of course the Klein-Gordon equation $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + m^2 \right) D = 0$ Let u and v be the positive and negative energy solutions of this Dirac equation in momentum space

$$u(q) = \left(2\omega(\omega+q) \right)^{-1/2} \begin{pmatrix} \omega+q \\ -im \end{pmatrix} \qquad v(q) = u(q)^*, \ \omega = \sqrt{m^2+q^2} \tag{66}$$

normalized to $|u_1|^2 + |u_2|^2 = |v_1|^2 + |v_2|^2 = 1$ and reducing in the zero mass limit to

$$u(q) \underset{m\to 0}{\longrightarrow} \begin{pmatrix} \theta(q) \\ -i\theta(-q) \end{pmatrix} \qquad v(q) \underset{m\to 0}{\longrightarrow} \begin{pmatrix} \theta(q) \\ i\theta(-q) \end{pmatrix}$$

as should be the case. The corresponding massive fermion field denoted by a superscript m is

$$D^m(r) = \frac{1}{\sqrt{2\pi}} \int dq \left(e^{iqr} u(q) A_q + e^{-iqr} v(q) B_q^\dagger \right) \tag{67}$$

The equal-time Wightman function for this field is an average computed in a vacuum state satisfying $A_q|0\rangle_m = B_q|0\rangle_m = 0$, and is found to be

$$\langle D_\alpha^m(r)\, D_\beta^{m\dagger}(r')\rangle_m = \frac{1}{2\pi}\int \frac{dq}{2\omega} e^{iq(r-r')} \begin{pmatrix} \omega+q & im \\ -im & \omega-q \end{pmatrix} \tag{68}$$

As $m \to 0$, $\rho = |r-r'|$ finite we find with $C = \frac{1}{2}e^\gamma$:

$$\langle D_\alpha^m(r)\, D_\beta^{m\dagger}(r')\rangle_m = \begin{pmatrix} \frac{i}{2\pi(\rho+i\epsilon)} & 0 \\ 0 & \frac{-i}{2\pi(\rho-i\epsilon)} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{im}{2\pi}\ln Cm\rho \\ \frac{im}{2\pi}\ln Cm\rho & 0 \end{pmatrix} + \dots \tag{69}$$

The first term is of course the mass zero value $\langle D_\alpha(r)\, D_\beta^\dagger(r')\rangle$ and the correction term is of order m ln m . It would seem that perturbation theory in m is in trouble. However the origin of the ln m factor lies in an infrared singularity. If we assume the existence of an infrared cutoff of order m in momentum space, or $\frac{1}{m}$ in configuration space, we should be able to recover from perturbation theory that to lowest order

$$\delta\langle D_\alpha(r)\, D_\beta^\dagger(r')\rangle = \begin{pmatrix} 0 & -\frac{im}{2\pi}\ln Cm\rho \\ \frac{im}{2\pi}\ln Cm\rho & 0 \end{pmatrix} \tag{69'}$$

Let us now see if this idea works. The interaction picture Gell-Mann-Low formula tells us that

$$\langle D_\alpha^m(r,0)\, D_\beta^{m\dagger}(r',0)\rangle_m = \frac{\langle T D_\alpha(r,0) D_\beta^\dagger(r',0) \exp\{-i\int d^2z\, \mathcal{H}_m(z)\}\rangle}{\langle \exp\{-i\int d^2z\, \mathcal{H}_m(z)\}\rangle} \tag{70}$$

On the right-hand side we use massless fields and a Minkowski space formula. Since we are computing an equal-time Wightman function which does not distinguish between real and imaginary time, this should not make any difference. The time evolution of the massless field is simply given by (39) so that the propagators are

$$\langle T D_1(x) D_1^+(y)\rangle = \frac{i}{2\pi} \frac{1}{x^1-y^1-(x^0-y^0)(1-i\epsilon)}$$
$$\langle T D_2(x) D_2^+(y)\rangle = \frac{-i}{2\pi} \frac{1}{x^1-y^1+(x^0-y^0)(1-i\epsilon)} \tag{71}$$

and $\langle T D_2(x)\, D_1^+(y)\rangle$ vanishes. With $\mathcal{H}_m$ given by (63), satisfying $\langle \mathcal{H}_m(z)\rangle = 0$, we find to lowest order

$$\delta\langle D_\alpha(r,0) D_\beta^+(r',0)\rangle = -m\int d^2z \langle T D_\alpha(r,0) D_\beta^+(r',0) \bar{D}(z)\gamma^5 D(z)\rangle . \tag{72}$$

Applying Wick's theorem we see that this implies

$$\delta \langle D_\alpha(r,0)\, D_\alpha^+(r',0)\rangle = 0 \tag{73}$$

in agreement with (69); while for instance

$$\delta\langle D_1(r,0) D_2^+(r',0)\rangle = m\int d^2z \langle T D_1(r,0) D_1^+(z)\rangle\langle T D_2(r',0) D_2^+(z)\rangle$$
$$= \frac{m}{(2\pi)^2}\int dz^1 \int dz^0 \frac{1}{\left(r-z^1+z^0(1-i\epsilon)\right)\left(z^1-r'+z^0(1-i\epsilon)\right)} \tag{74}$$

The z^0 integral is readily evaluated by contour integral methods and with $\rho = |r'-r|$ we obtain, in configuration space, the logarithmic infrared divergent integral:

$$\delta\langle D_1(r,0)\, D_2^+(r',0)\rangle = \frac{im}{2\pi}\int_0^\infty \frac{du}{u+\rho} . \tag{75}$$

This is exactly what was expected and we supply an infrared cutoff by replacing the upper limit of integration by $\frac{1}{Cm}$ where, of course, the constant C is here unknown. Then

$$\delta\langle D_1(r,0)\, D_2^+(r',0)\rangle = \frac{im}{2\pi}\int_0^{\frac{1}{Cm}} \frac{du}{u+\rho} = \frac{-im}{2\pi}\ln Cm\rho \tag{76}$$

Fortunately we thus reproduce the exact result given in (69). Thus mass perturbation works, at least to lowest order, with an infrared cutoff, and we feel confident to apply the same method to the Ising correlation function.

D. First Order Correction to the Correlation Function

In (58) we wrote the correlation function as an expectation value of a product of non-hermitian fields $\mathcal{N} e^{i\sqrt{\pi}\Phi(0)} \mathcal{N} e^{-i\sqrt{\pi}\Phi(\rho)}$. It is clear that we could have used their hermitian conjugates. Furthermore the formula would have been insensitive to a shift of Φ by a constant amount θ, reflecting the spurious chiral degeneracy, introduced by the use of a complex Dirac field. The latter arbitrariness is fixed by requiring the vacuum to give Φ a vanishing expectation value and using instead of (58) the equally good choice

$$\langle \sigma_0 \sigma_\rho\rangle^2 = 2 \langle \mathcal{N}\sin\sqrt{\pi}\,\Phi(0)\; \mathcal{N}\sin\sqrt{\pi}\,\Phi(\rho)\rangle \tag{77}$$

From (57) we see that the added terms do not contribute in the limit $\rho\to 0$. This choice however commits us to a choice of perturbing Hamiltonian $\mathcal{H}_m$ in terms of Φ. Coleman's work[3] enables one to translate the massive Thirring model into a sine-Gordon model. We have here the special case corresponding to a free massive Fermi field and thus in Coleman's notations the appropriate coupling constant satisfies $\beta^2 = 4\pi$. Thus to express

$$\mathcal{H}_m = -im\,\bar{D}\gamma^5 D = im\left(\Sigma^{(+)} - \Sigma^{(-)}\right)$$
$$\Sigma^{(+)} = D_1^+ D_2 \qquad \Sigma^{(-)} = \Sigma^{(+)\dagger} \tag{79}$$

in terms of Φ all that is necessary is to find the correct normalization. The Minkowskian time ordered products can be rotated in the complex time plane (rotation $-\frac{\pi}{2}$) to yield euclidean averages. For instance

$$\langle \Sigma^{(+)}(x)\,\Sigma^{(-)}(y)\rangle = \frac{1}{(2\pi)^2}\,\frac{1}{|x-y|^2}\;, \tag{80}$$

while for instance from (57)

$$\langle \mathcal{N} e^{i\sqrt{4\pi}\Phi(x)}\,\mathcal{N} e^{-i\sqrt{4\pi}\Phi(y)}\rangle = \frac{1}{|x-y|^2} \tag{81}$$

Similar relations can be written for strings of Σ's or exponentials of Φ. Thus we have the identification of $\Sigma^{(+)}(x)$ with $\frac{1}{2\pi}\mathcal{N} e^{i\sqrt{4\pi}\Phi(x)}$, up to a phase and thus adjusting the phase to $e^{i\pi/2}$, we find

$$\mathcal{H}_m = -\frac{m}{\pi}\,\mathcal{N}\cos\sqrt{4\pi}\,\Phi\;. \tag{82}$$

The choice of phase is dictated by a classical argument requiring the positivity of energy, and we assume for the moment $m > 0$. The corresponding Lagrangian for the Φ field reads, using Lorentz variables and renormalized fields:

$$\mathcal{L}(\Phi) = \frac{1}{2}:(\partial\Phi(x))^2: + \frac{m}{\pi}\mathcal{N}\cos\sqrt{4\pi}\,\Phi(x) + j(x)\,\mathcal{N}\sin\sqrt{\pi}\,\Phi(x)\;. \tag{83}$$

The source $j(x)$ is coupled to $\sin\sqrt{\pi}\,\Phi$, the effective field of interest and hence plays a role analogous to a magnetic field. From (55) we recall that the dimensions are

$$\begin{aligned} \dim\left(\mathcal{N}\cos\sqrt{4\pi}\,\Phi\right) &= 1 \\ \dim\left(\mathcal{N}\sin\sqrt{\pi}\,\Phi\right) &= 1/4 \quad , \end{aligned} \tag{84}$$

so that (83) is dimensionally correct but presupposes a vanishing dimensional parameter μ. The latter could possibly be related to a multiple of m.

The equivalent of the Gell-Mann-Low formula in euclidean space time is

$$\langle\sigma_0\sigma_\rho\rangle_m^2 = 2\,\frac{\langle \mathcal{N}\sin\sqrt{\pi}\,\Phi(0)\,\mathcal{N}\sin\sqrt{\pi}\,\Phi(\rho)\exp\left\{-\int d^2z\,\mathcal{H}_m(z)\right\}\rangle}{\langle\exp\left\{-\int d^2z\,\mathcal{H}_m(z)\right\}\rangle} \tag{85}$$

Since $\langle\mathcal{N}\cos\sqrt{4\pi}\,\phi\rangle = 0$, to lowest order

$$\langle\sigma_0\sigma_\rho\rangle_m^2 = \langle\sigma_0\sigma_\rho\rangle^2 + \delta\langle\sigma_0\sigma_\rho\rangle^2$$

$$\begin{aligned} \delta\langle\sigma_0\sigma_\rho\rangle^2 &= \frac{2m}{\pi}\int d^2z\,\langle\mathcal{N}\sin\sqrt{\pi}\,\Phi(0)\,\mathcal{N}\sin\sqrt{\pi}\,\Phi(\rho)\,\mathcal{N}\cos\sqrt{4\pi}\,\Phi(z)\rangle \\ &= -\frac{m}{2\pi}\,\rho^{1/2}\int d^2z\,\frac{1}{|z|\,|z-\rho|} \end{aligned} \tag{86}$$

In the last expression z and ρ are two-dimensional vectors. Using an infrared cutoff of order $1/Cm$ (for instance in the translational invariant form $\frac{1}{|z-x|}\rightarrow\frac{e^{-Cm|z-x|}}{|z-x|}$) we readily find

$$\delta\langle\sigma_0\sigma_\rho\rangle^2 = m\rho^{1/2}\ln Cm\rho = \langle\sigma_0\sigma_\rho\rangle^2 m\rho\ln Cm\rho \tag{87}$$

Consequently

$$\langle\sigma_0\sigma_\rho\rangle = \rho^{-1/4}\left(1+\frac{1}{2}m\rho\ln C|m|\rho+\cdots\right). \tag{88}$$

It turns out that (88) is also valid for $m<0$. We can see this as follows. In the absence of source and for m positive, the minimum of energy as derived from (82) and (83) is at $\Phi = 0$, and the source term drives it to a value in the vicinity of this point. This in fact justifies our choice of coupling to $\sin\sqrt{\pi}\,\Phi$ rather than say $\cos\sqrt{\pi}\,\Phi$. This was no innocent choice for the m ln m correction as it would have amounted to a change of sign. For $m<0$ this analysis fails. To recover a positive energy a finite shift $\Phi \to \Phi \pm \frac{\sqrt{\pi}}{2}$ is necessary. This we can realize by keeping the previous choice of $\mathcal{H}_m$ with m replaced by $|m|$, and shifting $\sin\sqrt{\pi}\,\Phi$ to $\cos\sqrt{\pi}\,\Phi$ (up to an irrelevant sign) which as we said changes the term $\frac{1}{2}|m|\rho \ln C|m|\rho$ into its opposite. Hence (88) holds for $m<0$ and coincides with the value quoted in Ref. 4.

It can also be noted that lowest order mass perturbation yields the correct singularity of the specific heat already obtained in (34). Indeed the denominator of (85) can be interpreted as

$$\exp\left\{L^2\, 2\,\Delta F(m)\right\} = \left\langle \exp\left\{-\int d^2z\ \mathcal{H}_m(z)\right\}\right\rangle \tag{89}$$

The factor L^2 is expected here from translational invariance while the factor 2 arises from the two non-interacting Ising systems. Thus to leading order

$$\begin{aligned}\Delta F &= \frac{1}{4L^2}\left(\frac{m}{\pi}\right)^2 \iint d^2z_1\, d^2z_2 \left\langle \mathcal{N}\cos\sqrt{4\pi}\,\Phi(z_1)\ \mathcal{N}\cos\sqrt{4\pi}\,\Phi(z_2)\right\rangle \\ &= \frac{m^2}{8\pi^2}\int \frac{d^2z}{|z|^2}\,.\end{aligned} \tag{90}$$

To make sense of this expression we need both an infrared cutoff $|z| < 1/C|m|$ and an ultraviolet cutoff $|z| > a$, a being the lattice spacing taken as unity. Thus

$$\Delta F = -\frac{m^2}{4\pi}\ln|m| + O(m^2), \tag{91}$$

in agreement with (34). This is a check on the consistency of this approach.

IV. CONCLUDING REMARKS

The derivation of the critical theory of the Ising two-dimensional model as described by the Lagrangian (83) has been straightforward even though here and there a sleight of hand was necessary. Actually if another method than mass perturbation is to be used, it would seem preferable to modify the renormalization prescription involved in the $\mathcal{N}$ symbol with its dimensional parameter μ . Perhaps as is suggested from the soliton theory one should keep μ finite and related to m in such a way that the soliton mass, identified with the fermion one, be equal to m. The complete calculation of the two-point function for any value of $m\rho$ given in Ref. 4 could then be interpreted as solving the quantization problem of the sine-Gordon theory for the particular coupling $\sqrt{4\pi}$. It seems likely that the soluble two-dimensional models of the Baxter type have a critical theory described by a Thirring model[6] or equivalently by a sine-Gordon equation with interaction proportional to $\cos\beta\Phi$. The leading corrections to their scaling behavior could therefore be computed. Also one should investigate higher-order correlation functions. It is unclear whether our trick of replacing an exponential by a sine function remains valid. It should also be remembered that higher correlation require unequal time calculations.

ACKNOWLEDGMENTS

One of the authors (C.I.) is happy to acknowledge the hospitality of the Fermi National Accelerator Laboratory, where this work was completed. He has benefited from stimulating conversations with M. Bander.

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