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Asymptotic Symmetry*

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ABSTRACT

We discuss the deep Euclidean behavior of Green's functions in models in which symmetries are broken spontaneously or by "soft" operators. Our proof that the Green's functions approach their symmetric values in the deep Euclidean region uses the Callan-Symanzik equation in a way which avoids relying on order by order power counting for the scalar tadpole insertions. The vacuum expectation value of the scalar field appears in the Callan-Symanzik equation as an additional coupling constant; the solution of the equation introduces a momentum dependent effective vacuum expectation value. We discuss theories with or without gauge invariance of the second kind.

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I. INTRODUCTION

The revival of interest in the renormalization group, particularly in the modern form of the Callan-Symanzik equations,[1] has generated a great deal of study of the asymptotic behavior of field theories at large spacelike momenta. An important physical motivation for investigating this apparently unphysical limit is the hope of understanding the behavior of weak and electromagnetic interaction of hadrons when the corresponding currents or gauge mesons are far off-shell. These studies probe possible field theory models of hadronic matter as well as high energy and higher-order effects of non-strong interactions.

The model under investigation involves some internal symmetry group and computations are carried out with the symmetry preserved. In the real world, however, many of these symmetries are broken and one wonders whether this implies any modification of the asymptotic behavior deduced in the symmetric limit. If the symmetry breaking term is sufficiently "soft", one expects from power counting arguments that it will have negligible effect in the deep Euclidean region.

In this paper, we confirm this effect when the symmetry breaking arises from the non-vanishing vacuum expectation value of a scalar field. Such an effect may arise either from an explicit term linear in the scalar fields in the Lagrangian density or from spontaneous breakdown. Our proof that the Green's functions approach their symmetric values in the deep Euclidean region uses the Callan-Symanzik

(C-S) equations in a way which avoids relying on order by order power counting for the scalar tadpole insertions. This procedure is necessary in a spontaneously broken symmetry theory, since in this case the Green's functions are in general not analytic in the vacuum expectation value of the scalar field. The vacuum expectation value of the scalar field appears in the C-S equations as an additional coupling constant; the solution of the equation introduces a momentum dependent effective vacuum expectation value (VEV).

The symmetric point corresponding to zero VEV is a fixed point of the effective VEV. For theories without gauge particles, where a non-zero VEV implies a Goldstone boson, positivity properties lead to the conclusion that this fixed point is unique and ultra-violet stable. For gauge theories in a general renormalizable gauge, these positivity conditions are lacking and such an unequivocal conclusion is not available. In this case we have calculated in the one loop approximation. To this order, we show that the theory approaches its symmetric limit for all choices of the gauge parameter if the effective gauge coupling is sufficiently small. This conclusion holds, therefore, when the theory is "asymptotically free."[2] We do not, however, consider the problems of maintaining asymptotic freedom when Higgs mesons are added to a gauge theory.

We prove our results by considering simple prototype Lagrangians. The generalizations to more complicated theories with additional couplings constants seem straightforward. In the next section we discuss the scalar field model where spontaneous breakdown implies a Goldstone boson. In the third section, we consider a gauge theory with Higgs scalars added. Finally we offer some concluding remarks and ideas for extensions of this work.

II. A MODEL WITH GLOBAL INVARIANCE

As a prototype of a model with a Goldstone boson in the broken symmetry limit, we consider the field theory of a massive complex scalar field with a $\lambda_0 |\phi|^4$ interaction. The Callan-Symanzik equations for the renormalized irreducible proper vertex functions for a symmetric solution with vanishing vacuum expectation value of ϕ can be written (The complex field ϕ may be thought of as a two-dimensional real vector in an isospace, so that $\phi = \phi_1 + i\,\phi_2$. The subscript i in p_i includes the specification of the component in this isospace):

$$[m^2 \frac{\partial}{\partial m^2} + \beta(\lambda)\frac{\partial}{\partial\lambda} - (n/2)\,\gamma(\lambda)]\Gamma(p_1, \ldots, p_n; m^2, \lambda)$$

$$= \Gamma_\Delta(0; p_1, \ldots, p_n; m^2, \lambda) \tag{1}$$

where m^2 and λ are the physical mass and coupling constant defined by

$$\Gamma(p, -p)\Big|_{p^2 \to m^2} = (p^2 - m^2)$$

$$\Gamma(p_1, p_2, p_3, p_4)\Big|_{\text{sym. pt.}} = -\lambda$$

The symmetric point is defined by $p_i \cdot p_j = m^2/3\ [4\delta_{ij} - 1]$.

The vertex functions on the right hand side of the equation have one zero-momentum external insertion of the mass operator $|\phi|^2$ whose strength is normalized by the convention

$$\Gamma_\Delta(0,p,-p)\Big|_{p^2=m^2} = -m^2$$

The C-S equation for the generating functional of the vertex function will be useful and is constructed from

$$\Gamma(\Phi) = \sum_n (1/n!) \int \prod_{i=1}^{n} dx_i\, \Phi(x_i)\, \Gamma(x_1, \ldots, x_n; m^2 \lambda) \tag{2}$$

Since the differential operator in (1) does not act on the momenta, the equation holds also in coordinate space. Using the functional analogue of

$$Z \frac{d}{dZ} Z^n = n Z^n$$

the desired equation can be written

$$\left[m^2 \frac{\partial}{\partial m^2} + \beta(\lambda)\frac{\partial}{\partial \lambda} - \gamma(\lambda)/2 \int dx\, \Phi(x) \frac{\delta}{\delta \Phi(x)} \right] \Gamma(\Phi) = \Gamma_\Delta(0;\Phi) \tag{3}$$

$\Gamma_\Delta(0,\Phi)$ is defined analogously to $\Gamma(\Phi)$.

The n-point vertex function for a solution with non-vanishing vacuum expectation value of ϕ, say $\langle\phi\rangle_0 = v$, is given by the n-th functional derivative of Γ evaluated at $\Phi = v$. In general, this is the solution for the theory with an external source of the scalar field $J \neq 0$, or a linear term in ϕ added to the original Lagrangian density. The possibility of $\langle\phi\rangle_0 \neq 0$ for no source, $J = 0$, corresponds to spontaneous breakdown. The values of the parameters for which this is possible can be determined from the superpotential.[3] The spontaneous breakdown condition

$$\left.\frac{\delta\Gamma}{\delta\,\Phi}\right|_{\Phi = v} = J = 0$$

leads to a relation between v, m^2, λ. In deriving the C-S equations for $v \neq 0$, however, we consider v, m^2, λ as independent parameters. Each set corresponds to some determinable value of J.

To obtain the C-S equation for the vertex functions of the solution with $\Phi = v$, we write $\tilde{\Phi} = \Phi + v$ and note that

$$\int dx\ \Phi(x)\ \frac{\delta}{\delta\,\Phi(x)}\,\Gamma\,(\Phi = \tilde{\Phi} + v) = \left[\int dx\ \tilde{\Phi}(x)\,\frac{\delta}{\delta\,\tilde{\Phi}(x)} + v\,\frac{\partial}{\partial v}\,\right]\Gamma \qquad (3)$$

Then taking the n-th functional derivative of Γ with respect to $\tilde{\Phi}$ $\tilde{\Phi} = 0$ in Eq. (2) gives the C-S equation for the broken symmetry

$$\left[m^2\frac{\partial}{\partial m^2} + \beta(\lambda)\,\frac{\partial}{\partial\lambda} - \gamma(\lambda)\,\frac{1}{2}(n + v\frac{\partial}{\partial v}\,)\right]\Gamma\,(p_1, \ldots, p_n, m^2, \lambda v)$$

$$- \Gamma_{\Delta}\,(0;p_1, \ldots, p_n, m^2, \lambda, v) \qquad (4)$$

In order to admit the case of spontaneous breakdown of the symmetry, we must allow the possibility that $m^2 < 0$.[4] Equations (3) and (4) are not affected even in such cases.

For nonexceptional momenta, the inhomogeneous term can be dropped in the deep Euclidean region by the usual power counting theorem of Weinberg. By ordinary dimensional counting, we can write the vertex function as

$$\Gamma = s^{(4-n)/2} F^n (s/m^2, v/m, \lambda, \{u_{ij}\}) \tag{5}$$

where $s = -\sum_i p_i^2$, $u_{ij} = \frac{p_i p_j}{s}$.

The deep Euclidean region is defined by $s \to \infty$, u_{ij} fixed. In this limit the C-S equation reduces to

$$\{m^2 \frac{\partial}{\partial m^2} + \beta(\lambda)\frac{\partial}{\partial \lambda} - [\gamma(\lambda)/2]\, v\frac{\partial}{\partial v} - (n/2)\gamma(\lambda)\}\, F^n_{AS} = 0 \tag{6}$$

Upon introducing the variables $t = \ln(s/m^2)$, $x = v/m$, Eq. (6) becomes

$$[-\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \lambda} - (x/2)(1+\gamma)\frac{\partial}{\partial x}]\, F^n_{AS} = (n/2)\gamma\, F^n_{AS} \quad . \tag{7}$$

In this form v or x plays the role of a second coupling constant.[5] To solve, introduce the effective coupling constants $\bar\lambda(\lambda, t)$, $\bar{x}(x, \lambda, t)$ by

$$\begin{aligned} \frac{\partial\bar\lambda}{\partial t} &= \beta(\bar\lambda), \qquad \bar\lambda(\lambda, 0) = \lambda \\ \frac{\partial \bar{x}}{\partial t} &= -(\bar{x}/2)[1+\gamma(\bar\lambda)], \quad \bar{x}(x, \lambda, 0) = x \end{aligned} \tag{8}$$

$\bar{x}$ has a fixed point at $x = 0$; the symmetric solution remains symmetric when the momenta are continued to the deep Euclidean region. Furthermore, $\bar{x} = 0$ is the only fixed point and is ultraviolet stable if $1 + \gamma(\lambda) > 0$. For a theory with a positive metric, it follows from the Källen-Lehmann representation for the scalar two-point function that $\gamma(\lambda) > 0$ and the necessary condition above obviously holds. That is,

$$\lim_{t \to \infty} \bar{x}(x, \lambda, t) = 0$$

which implies that the vertex function approaches its symmetric value. To illustrate this more explicitly we note that Eq. (8) gives $\bar{\lambda}(\lambda, t)$ as the same function as in the symmetric solution, and that the solution of (8) is

$$\bar{x}(x, \lambda, t) = \exp \left\{ -1/2 \left\{ t + \int_0^t \gamma\left[\bar{\lambda}\left(\lambda, t'\right)\right] dt' \right\} \right\} \tag{9}$$

which shows that $\bar{x} \to 0$ faster than $1/\sqrt{s}$.

The asymptotic vertex function can be written

$$F^n_{AS} = \zeta^n\left(\bar{\lambda}, \bar{x}, \{u\}\right) \exp\left\{ -n/2 \int_0^t \gamma\left[\bar{\lambda}\left(\lambda, t'\right)\right] dt' \right\} \tag{10}$$

The form of ζ^n is not determined from the homogeneous equation, but continuity demands that $\zeta^n\left(\bar{\lambda}, 0, \{u\}\right)$ corresponds to the symmetric solution. If λ has an ultraviolet fixed point at $\lambda = \lambda_1$, corresponding to a first order zero of $\beta(\lambda_1)$, the asymptotic solution becomes

$$F^n_{AS} = \zeta^n \left(\lambda_1, x\, e^{-[t/\lambda]\,[1+\gamma(\lambda_1)]} K, \{u\} \right)$$
$$\times\ (s/m^2)^{-n\gamma(\lambda_1)/2}\, K^n \tag{11}$$

with

$$K = \exp\left\{ -1/2 \int_\lambda^{\lambda_1} \left[\gamma(\lambda') - \gamma(\lambda_1)\right] d\lambda' / \beta(\lambda') \right\}$$

III. A MODEL WITH LOCAL GAUGE INVARIANCE

To extend these ideas to spontaneously broken gauge theories, we consider a theory of vector mesons and scalars invariant under local transformations of a gauge group G. The Lagrangian is

$$\mathscr{L} = 1/2 \,|\, \partial_\mu \phi - ig\, T^\alpha \phi A^\alpha_\mu \,|^2 - \left(m_0^2/2\right) |\phi|^2 - \left(\lambda_0/4\right) |\phi|^4$$

$$+ \mathscr{L}_{GAUGE}$$

where $(T^\alpha)_{ab}$ is a real representation of the group G; $T^\alpha = -(T^\alpha)^t$ (We assume for simplicity that the representation content of the scalars admits only one quartic coupling.)

In the symmetric limit we renormalize the scalar propagator by the convention

$$\lim_{p^2 \to \mu^2} \Delta^{-1}(p^2) = p^2 - M^2 + 0[(p^2 + \mu^2)^2]$$

and

$$\lim_{p^2 \to \mu^2} \Delta_{\mu\nu}(p^1) = -g_{\mu\nu} \frac{1}{\mu^2} + \text{gauge dependent term}$$

and other renormalization parts at some Euclidean points characterized in terms of μ^2. The unrenormalized Green's functions are independent of the value μ^2; this leads to the Gell-Mann-Low equation[6]

$$\Big[\mu^2 \frac{\partial}{\partial \mu^2} + \beta^1_g \frac{\partial}{\partial g} + \beta^1_\lambda \frac{\partial}{\partial \lambda} - (n_v/2)\gamma^1_v - (n_s/2)\gamma^1_s$$

$$- 2\gamma^1_v \frac{\partial}{\partial \alpha} \Big] \Gamma^{n_v, n_s}(k_1, \ldots, k_{n_v}, p_1, \ldots, p_{n_s}; m^2, \mu^2, g, \lambda) = 0$$

where n_v = number of external vector lines

n_s = number of external scalar lines

$\beta_g^1 = \partial g/\partial \ln \mu^2$

$\beta_\lambda^1 = \partial \lambda/\partial \ln \mu^2$

$\gamma_v^1 = \partial Z_v/\partial \ln \mu^2$

$\gamma_s^1 = \partial Z_s/\partial \ln \mu^2$

and $\alpha = \alpha_0/Z_v$ is the coefficient of the longitudinal term in the renormalized vector meson propagator.

External scalar mass insertions gives the C-S equation

$$[m^2 \frac{\partial}{\partial m^2} + \beta_g^2 \frac{\partial}{\partial g} + \beta_\lambda^2 \frac{\partial}{\partial \lambda} - (n_v/2)\gamma_v^\alpha - (n_s/2)\gamma_s^2$$
$$- 2\gamma_v^2 \frac{\partial}{\partial \alpha}]\Gamma^{n_v, n_s} = \Gamma_\Delta^{n_v, n_s} \tag{13}$$

where

$$\beta^2 g = \partial g/\partial \ln m^2;$$
$$\beta_\lambda^2 = \partial \lambda/\partial \ln m^2;\ \text{etc.}$$

Adding the two equations gives

$$[m^2 \frac{\partial}{\partial m^2} + \mu^2 \frac{\partial}{\partial \mu^2} + \beta_g \frac{\partial}{\partial g} + \beta_\lambda \frac{\partial}{\partial \lambda} - (n_v/2)\gamma_v - (n_s/2)\gamma_s$$
$$- \alpha \gamma_v \frac{\partial}{\partial \alpha}]\Gamma^{n_v, n_s} = \Gamma_\Delta^{n_v, n_s} \tag{14}$$

with

$$\beta_g = \beta_g^1 + \beta_g^2,\ \text{etc.}$$

and

$$\gamma_v = (\mu^2 \frac{\partial}{\partial \mu^2} + m^2 \frac{\partial}{\partial m^2})\ln Z_v\ ;\ \text{etc.}$$

At the one loop level we can regulate the theory by introducing an explicit ultraviolet cut-off Λ^2. To this order, then,

$$\gamma_v = - \partial \ln Z_v / \partial \ln \Lambda^2, \text{ etc.}$$

[We take $\lambda \sim 0(g^2)$ so that expansion in $g^2 \sim \lambda$ is equivalent to an expansion in the number of loops.]

When the scalar field has a nonvanishing vacuum expectation value v, the modified Callan-Symanzik equations are obtained exactly as in the preceding section. Equation (14) is altered by the addition of a term

$$-(\gamma_s/2) v \frac{\partial}{\partial v} \Gamma^{n_v, n_s}$$

to the left hand side.

By dimensional power counting

$$\Gamma^{n_v, n_s} = s^{(4-n_v-n_s)/2} F^{n_v, n_s}(s/m^2, \mu^2/m^2, x=v/m, g, \lambda, u_{ij})$$

and the C-S equation in the deep Euclidean region becomes

$$[- \frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + \beta_\lambda \frac{\partial}{\partial \lambda} - (x/2)(1 + \gamma_s) \frac{\partial}{\partial x} - \alpha \gamma_v \frac{\partial}{\partial \alpha}$$

$$- (n_v/2)\gamma_v - (n_s/2)\gamma_s] \; F_{AS}^{n_v, n_s} = 0 \qquad (1 \quad (15)$$

with

$$s = - \sum_i p_i^{\;2} - \sum_i k_i^{\;2}, \; t = \ln (s/m^2)$$

Introducing effective coupling constants, $\bar{g}, \bar{\lambda}, \bar{x}, \bar{\alpha}$ as in Sec. II, the sufficient condition for asymptotic symmetry is

$$\lim_{t \to \infty} \gamma_s (\bar{g}, \bar{\lambda}, \bar{\alpha}, t) > -1 \qquad (16)$$

Since the vector theory in a general renormalizable gauge is realized in a Hilbert space of indefinite metric, we cannot appeal to the same positivity properties as before. Note, however, that the equations for $\bar{g}, \bar{\lambda}, \bar{\alpha}$ are independent of $\bar{x}$ and that γ_V, γ_S are functions of $\bar{g}, \bar{\lambda}, \bar{\alpha}$ only. Therefore, the power behavior in s of the Green's functions is the same for the symmetric solution. Only the coefficient function denoted by ζ in the previous section might be affected. (We assume that ζ has a finite limit as $t \to \infty$).

We have no general proof of the inequality of Eq. (16). However, we have studied the problem in the one loop expansion of pertubation theory.

The one loop contribution to Z_S is

$$Z_S = 1 + \frac{g^2}{16\pi^2} \sum_\beta (-T^\beta T^\beta)_{\alpha\alpha}(1+\alpha) \ell n\, (m^2/\Lambda^2) + 0\,(g^4)$$

and

$$\gamma_S = \frac{g^2}{16\pi^2} \sum_\beta (-T^\beta T^\beta)_{aa}\,(1+\alpha) + 0\,(g^4) \tag{17}$$

Note that $(-T^\beta T^\beta)_{aa} > 0$. The effective gauge parameter is determined from

$$\frac{\partial \bar{\alpha}}{\partial t} = -\bar{\alpha}\ \gamma_V(\bar{g}, \bar{\lambda}, \bar{\alpha})$$

The one loop contribution to γ_V is

$$\gamma_V = -\frac{g^2}{16\pi^2}(13/3 - \alpha)C_2(G) + \frac{g^2}{16\pi^2}\frac{1}{3}\ T_r(-T^\alpha T^\alpha) \tag{18}$$

The first term comes from the vector meson (and ghost) loop contributions to the gauge boson vacuum polarization and the second term from the Higgs scalar loop. $C_2(G)$ is the quadratic Casimir operator of the adjoint representation. $Tr(-T^\alpha T^\alpha) > 0$ and its value depends on the group representation realized on the Higgs scalars. There are two general possibilities depending on the sign of $13\ C_2(G) - Tr(-T^\alpha T^\alpha)$.

When this term is positive $-\alpha\gamma_V$ has the general shape shown in Fig. 1a. $\alpha = 0$ is a fixed point but is ultraviolet instable. On the other hand,

$$\alpha_1 = \frac{g^2}{48\pi^2}[13\ C_2(G) + Tr(T^\alpha T^\alpha)]$$

is clearly an ultraviolet stable fixed point for $\alpha > 0$. For $\alpha < 0$, $\bar{\alpha}$ has a discontinuous behavior. $\bar{\alpha}$ reaches $-\infty$ at a particular finite value of t, jumps to $+\infty$, and then approaches α_1 from above. That is, α_1 is an ultraviolet attractor for all $\alpha \neq 0$.

When there are enough Higgs scalars so that $\alpha_1 < 0$, the shape of $-\alpha\gamma_V$ is as shown in Fig. 1b. In this case, $\alpha = 0$ is an ultraviolet attractor for all $\alpha \neq \alpha_1$.

It follows that except for $\alpha = \alpha_1 < 0$, the asymptotic effective gauge parameter is positive semi-definite, i.e.,

$$\lim_{t\to\infty} \bar{\alpha} \geq 0$$

Hence, from (17) $\lim_{t \to \infty} \gamma_s > 0$, $\ln \bar{x} = 0$ as shown in Sec. II, and the Green's functions approach their symmetric values in the asymptotic limit.

In the exceptional case $\alpha = \alpha_1 < 0$, γ_s can be negative if $\alpha_1 < -1$. As long as g^2 is sufficiently small, in particular, if the theory is asymptotically free, $\lim_{t \to \infty} \alpha_1(g)$ will be absolutely less than 1, and we will have $\lim_{t \to \infty} \gamma_s > -1$. We have shown that this is a sufficient condition for symmetric Green's functions in the deep Euclidean region.

IV. DISCUSSION

The details of our derivations apply to the case of spontaneous symmetry breaking or to symmetry breaking due to a term in the Lagrangian density linear in the scalar field(s). We expect, however, that whenever the symmetry breaking is sufficiently "soft", it will have negligible effect in the deep Euclidean regime. It seems to us likely that the criteria for "softness" is that the symmetry breaking be due to a generalized mass term of free field dimension less than four, in the langauge of K. Wilson.[8]

In fact, S. Weinberg has recently treated fermion mass terms in a way related to the development here.[9] In his development, the mass plays the role of a coupling constant in the renormalization group equations. The effective mass tends to zero as $s \to \infty$ if the anomalous dimension of the mass operator exceeds minus one.

A complete knowledge of which types of symmetry breaking becomes negligible in the high momentum limit would be useful in devising experimental tests of the underlying symmetry of the theory, as well as in justifying the theoretical simplification of computing asymptotic behavior in a completely symmetric model. For example, asymptotic equality of Green's functions of different weak and electromagnetic currents in the deep Euclidean region can lead to sum rules among the corresponding spectral functions.

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FIGURE CAPTIONS

Fig. 1a $-\alpha\gamma_v$ vs. α when $13C_2(G) + \mathrm{Tr}(T^\alpha T^\alpha) > 0$.

Fig. 1b $-\alpha\gamma_v$ vs. α when $13C_2(G) + \mathrm{Tr}(T^\alpha T^\alpha) < 0$.

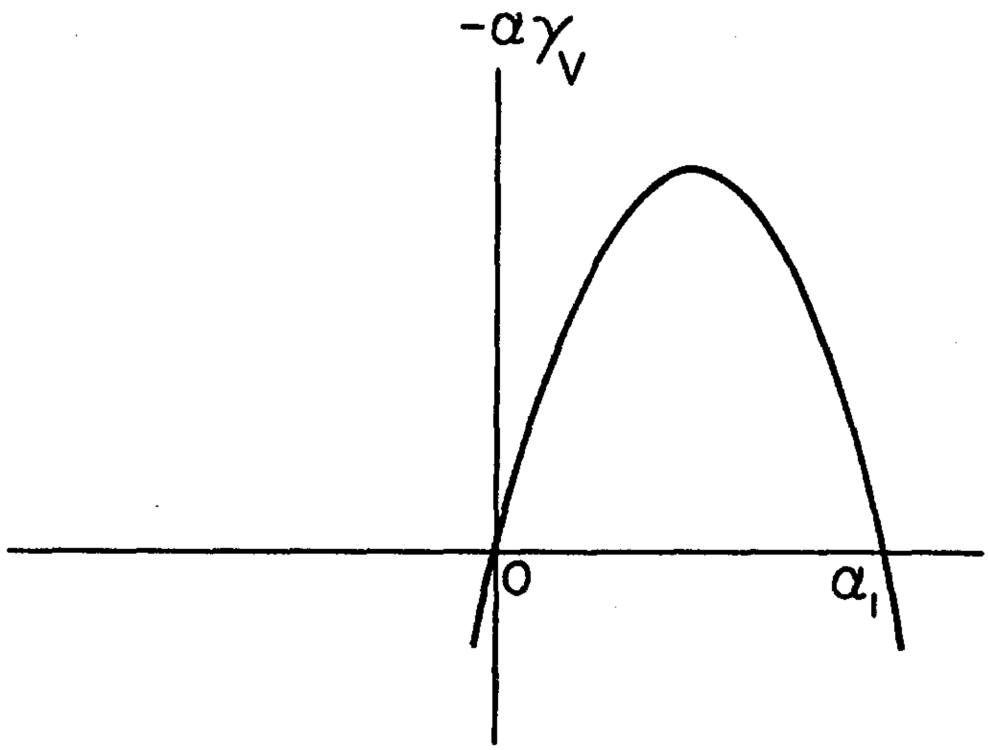

Figure 1a

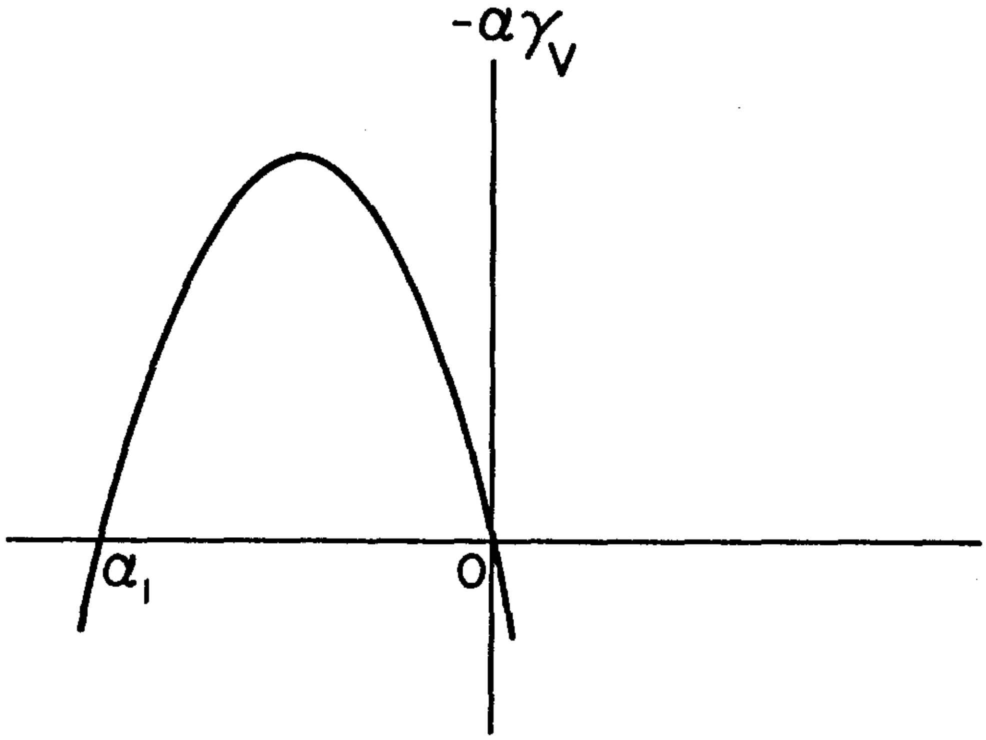

Figure 1b