Massless, Euclidean Quantum Electrodynamics on the 5-Dimensional Unit Hypersphere

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ABSTRACT

We show that the Feynman rules for vacuum polarization calculations and the equations of motion in massless, Euclidean quantum electrodynamics can be transcribed, by means of a stereographic mapping, to the surface of the 5-dimensional unit hypersphere. The resulting formalism is closely related to the Feynman rules, which we also develop, for massless electrodynamics in the conformally-covariant O(5, 1) language. The hyperspherical formulation has a number of apparent advantages over conventional Feynman rules in Euclidean space: It is manifestly infrared-finite, and it may permit the development of approximation methods based on a semiclassical approximation for angular momenta on the hypersphere. The finite-electron-mass, Minkowski-space generalization of our results gives a simple formulation of electrodynamics in (4, 1) de Sitter space.

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1. INTRODUCTION

Conformal invariance in quantum field theory has attracted renewed interest recently, because of its connection with problems of asymptotic high energy behavior.[1] Important results on leading light-cone singularities, for example, have been obtained by the use of conformal invariance.[2] Another question to which conformal invariance is relevant is the study of eigenvalue conditions imposed by requiring renormalization constants to be finite.[3] To see this, let us consider the single-fermion-loop vacuum polarization diagrams in spin-$\frac{1}{2}$ quantum electrodynamics, illustrated in Fig. 1. If we work in coordinate space with separated points x, x' we can freely pass to the zero-fermion-mass, or conformal limit. In this limit, however, the structure of the vacuum polarization is unique,[2] and hence the sum of diagrams in Fig. 1(a) must be proportional to the lowest order vacuum polarization tensor in Fig. 1(b),

$$\pi_{\mu\nu}(x,x';\alpha) = -3\pi F^{[1]}(\alpha)\pi^{(0)}_{\mu\nu}(x,x') \quad . \tag{1}$$

When Eq. (1) is Fourier transformed to momentum space, using current conservation in the usual fashion to eliminate the quadratic divergence, the function $F^{[1]}(\alpha)$ appears as the coefficient of the logarithmically divergent term. Requiring the photon wave function renormalization Z_3 to be finite then imposes the eigenvalue condition $F^{[1]}(\alpha) = 0$.[4]

Our aim in the present paper is to study reformulations of massless electrodynamics which are made possible by its invariance under conformal transformations, with the goal of developing methods which may allow one to calculate or approximate the function $F^{[1]}$ appearing in Eq. (1). Because the singularity structure in x and x' is not of interest (it is just that of the lowest order vacuum polarization), we make the Dyson-Wick rotation to a Euclidean metric at the outset. Thus we deal with massless, Euclidean quantum electrodynamics. Our principal result is that the Feynman rules for vacuum polarization calculations and the equations of motion in this theory can be simply rewritten in terms of equivalent rules and equations of motion on the surface of the 5-dimensional unit hypersphere. In Section 2 we state the 5-dimensional rules and verify by explicit transformation that they are equivalent to the usual rules in Euclidean coordinate space (x-space). We also construct and verify a 5-dimensional formulation of the Maxwell equations and the equation of current conservation, and discuss the physical meaning of rotations and inversions on the hypersphere. In Section 3 we discuss massless, Euclidean quantum electrodynamics in the manifestly conformal-covariant O(5,1) language. We develop the Feynman rules in this formalism, explore some of their peculiar features, and show that they are related by a simple projective transformation to the rules on the 5-dimensional hypersphere. In Section 4 we discuss possible generalizations and applications of our results. We point out that the finite-electron-mass, Minkowski-space extension of our hyperspherical results gives a simple formulation of electrodynamics in (4,1) de Sitter space. The electron wave equation which we use is just the de Sitter space equation originally proposed by Dirac,[5] but our treatment of the Maxwell equations is an improvement over that of Dirac, and does not require the imposition of homogeneity conditions. There are a number of calculational advantages of the hyperspherical formulation of electrodynamics over the usual Feynman rules in Euclidean space. First, because the surface of the hypersphere is a bounded domain, the calculation of vacuum polarization diagrams in the 5-dimensional formalism is manifestly infrared-finite. Second, because the wave operators on the hypersphere are constructed from angular momentum operators, there appears to be the possibility of making semiclassical approximations when virtual angular momentum quantum numbers are large compared to unity. This contrasts sharply with the situation in Euclidean space, where there is no natural distance or momentum scale which distinguishes regions where one can approximate the wave operator.

2. 5-DIMENSIONAL FORMALISM

In this section we set out the 5-dimensional formalism and verify, by explicit transformation, its equivalence to the usual rules in x-space. Sections 2(a)-2(c) contain a summary of the five-dimensional Feynman rules and equations of motion, while in Secs. 2(d) and 2(e) we discuss the transformation to x-space and the interpretation of symmetries on the hypersphere.

2(a) Summary of Feynman Rules on the Hypersphere

In writing down the 5-dimensional rules and comparing them with their Euclidean counterparts, we adhere to the following conventions and notation.[6] Five-dimensional unit vectors are denoted by $\eta_1, \eta_2, \ldots$; 5-dimensional vector indices are indicated by lower case Roman letters

$a, b, \ldots$ which take the values $1, \ldots, 5$, and the 5-dimensional metric is the Euclidean metric δ_{ab}. Similarly, ordinary 4-dimensional vectors are denoted by $x_1, x_2, \ldots$, with vector indices $\mu, \nu, \ldots$ taking the values $1, \ldots, 4$ and with a 4-dimensional Euclidean metric $\delta_{\mu\nu}$. The usual 4x4 Dirac γ-matrices are taken to satisfy a Euclidean Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu} \tag{2}$$

and are all hermitian; explicit representations for these matrices are well known. In writing the 5-dimensional rules we need, instead of the γ's, a set of five hermitian 8x8 matrices α_a satisfying the Clifford algebra

$$\{\alpha_a, \alpha_b\} = 2\,\delta_{ab} \quad . \tag{3}$$

In terms of the γ-matrices and the Pauli spin matrices $\tau_{1,2,3}$, an explicit representation of the α-matrices is

$$\alpha_\mu = \gamma_\mu \tau_1 \quad , \quad \alpha_5 = \tau_3 \quad ; \tag{4}$$

since the matrix

$$\alpha_6 = \tau_2 \tag{5}$$

satisfies $\alpha_6^2 = 1$ and anticommutes with the α_a, the trace of an odd number of α-matrices vanishes. Physical quantities such as the electromagnetic current, vector potential, etc. will be denoted by capital letters $(J_a, A_a, \ldots)$ in 5-dimensional space and by lower case letters $(j_\mu, a_\mu, \ldots)$ in Euclidean space. We let $\int d^4x = \int dx_1 dx_2 dx_3 dx_4$ denote the integration of x over Euclidean space and we similarly let $\int d\Omega_\eta$ denote the integration of η over the surface of the 5-dimensional hypersphere. Finally, we use tr_4 and tr_8 to denote respectively the trace over the

γ-matrices and the α-matrices.

The connection between the 5-dimensional coordinate η describing a space-time point and its Euclidean equivalent x is given by the stereographic mapping[7]

$$x_\mu = \kappa^{-1}\eta_\mu \quad , \quad \kappa = 1 + \eta_5 \quad , \tag{6a}$$

with the inverse transformation

$$\eta_\mu = \frac{2x_\mu}{1+x^2} \quad , \quad \eta_5 = \frac{1-x^2}{1+x^2} \quad . \tag{6b}$$

The 5-dimensional electromagnetic current J_a, which satisfies the constraint equation

$$\eta \cdot J = \eta_a J_a = 0 \tag{7}$$

is mapped into the usual electromagnetic current j_μ by

$$\kappa^{-3} j_\mu = J_\mu - x_\mu J_5 \quad , \tag{8a}$$

with the inverse transformation

$$J_\mu = \kappa^{-3} j_\mu - \kappa^{-2} x_\mu x \cdot j \quad , \quad J_5 = -\kappa^{-2} x \cdot j \quad . \tag{8b}$$

We can now state the 5-dimensional Feynman rules with, for comparison, their Euclidean counterparts:

	5-dimensional	Euclidean
electron propagator	$\frac{-i}{\pi^2} \frac{\frac{1}{2}(\alpha \cdot \eta_1 - 1)\frac{1}{2}(\alpha \cdot \eta_2 + 1)}{(\eta_1 - \eta_2)^4}$	$\frac{-i}{2\pi^2} \frac{\gamma \cdot (x_1 - x_2)}{(x_1 - x_2)^4}$
photon propagator	$\frac{1}{4\pi^2} \frac{\delta_{ab}}{(\eta_1 - \eta_2)^2}$	$\frac{1}{4\pi^2} \frac{\delta_{\mu\nu}}{(x_1 - x_2)^2}$ + gauge terms
electron-photon vertex	$ie\alpha_a \equiv \frac{1}{2} ie[\alpha \cdot \eta_1, \alpha_a]$	$ie\gamma_\mu$
each closed fermion loop	$-\mathrm{tr}_8$	$-\mathrm{tr}_4$
each virtual coordinate integration	$\int d\Omega_\eta$	$\int d^4x$

(9)

(The two indicated forms of 5-dimensional electron-photon vertex are equal when sandwiched between electron propagators.)

The equivalence of the two sets of Feynman rules for vacuum polarization (closed-fermion-loop) calculations is demonstrated explicitly in Sec. 2(d) below.

2(b) Photon Propagator Equation and Maxwell Equations

To write the wave equation satisfied by the photon propagator on the hypersphere we introduce the (antihermitian) angular momentum operator

$$L_{ab} = \eta_a \frac{\partial}{\partial \eta_b} - \eta_b \frac{\partial}{\partial \eta_a} \quad . \tag{10}$$

When several coordinates $\eta_1, \eta_2, \ldots$ are present we denote the angular momentum acting at η_1 by L_1, so $(L_1)_{ab} = (\eta_1)_a \, \partial/\partial(\eta_1)_b - (\eta_1)_b \, \partial/\partial(\eta_1)_a$, etc. In this notation the photon propagator equation takes the form

$$(L_1^{\,2}-4)\,\frac{1}{(\eta_1-\eta_2)^2} = -\,8\pi^2\,\delta_S(\eta_1-\eta_2) \quad , \tag{11}$$

where δ_S is the hyperspherical delta-function satisfying

$$\int d\Omega_{\eta_1} f(\eta_1)\delta_S(\eta_1-\eta_2) = f(\eta_2) \tag{12}$$

for arbitrary f. The constant multiplying the δ_S-function in Eq. (11) can be verified by integrating Eq. (11) over the hypersphere,

$$\begin{aligned}\int d\Omega_1 (L_1^{\,2}-4)\,\frac{1}{(\eta_1-\eta_2)^2} &= -\,4\int d\Omega_1\,\frac{1}{(\eta_1-\eta_2)^2}\\ &= -4\left[\frac{\int_{-1}^{1} d\mu(1-\mu^2)\,\frac{1}{2(1-\mu)}}{\int_{-1}^{1} d\mu(1-\mu^2)}\right]\int d\Omega_1\\ &= -4\,\left[\frac{1}{4/3}\right]\,\frac{8\pi^2}{3} = -\,8\pi^2 \quad ,\end{aligned} \tag{13}$$

where we have written $\mu = \eta_1\cdot\eta_2$ and used the fact that

$$\int d\Omega_1 = \int_{-1}^{1} d\mu(1-\mu^2)\times \text{azimuthal integrations} \quad . \tag{14}$$

Eq. (11) can also be verified from the expansion of $(1-\mu)^{-1}$ in terms of Gegenbauer polynomials $C_n^{3/2}(\mu)$,[8]

$$\frac{1}{1-\mu} = \sum_{n=0}^{\infty} \frac{(2n+3)C_n^{3/2}(\mu)}{(n+1)(n+2)} \quad . \tag{15}$$

Using the relation

$$\sum_m Y_{nm}(\eta_1)Y^*_{nm}(\eta_2) = \frac{2n+3}{8\pi^2}\;C_n^{3/2}(\eta_1\cdot\eta_2) \quad , \tag{16}$$

where the $Y_{nm}(\eta)$ are orthonormalized hyperspherical harmonics, Eq. (15) becomes

$$\frac{1}{(\eta_1-\eta_2)^2} = 4\pi^2 \sum_{n=0}^{\infty} \sum_m \frac{Y_{nm}(\eta_1) Y^*_{nm}(\eta_2)}{(n+1)(n+2)} \quad . \tag{17}$$

Then, using the differential equation for the hyperspherical harmonics

$$L_1^2 \, Y_{nm}(\eta_1) = - 2n(n+3) \, Y_{nm}(\eta_1) \quad , \tag{18}$$

we find from Eq. (17) that

$$\begin{aligned} (L_1^2-4) \frac{1}{(\eta_1-\eta_2)^2} &= - 8\pi^2 \sum_{n=0}^{\infty} \sum_m Y_{nm}(\eta_1) \, Y^*_{nm}(\eta_2) \\ &= - 8\pi^2 \, \delta_S(\eta_1-\eta_2) \quad , \end{aligned} \tag{19}$$

in agreement with Eq. (11).

To write the Maxwell equations on the hypersphere we introduce the electromagnetic potential 5-vector A_a, which satisfies the constraint

$$\eta \cdot A = 0 \tag{20}$$

and is related to the electromagnetic potential a_μ in Euclidean space by

$$\kappa^{-1} a_\mu = A_\mu - x_\mu A_5 \quad . \tag{21}$$

The electromagnetic field-strength is described by the totally antisymmetric rank-three tensor

$$F_{abc} = L_{ab} A_c + L_{bc} A_a + L_{ca} A_b \quad , \tag{22}$$

which is dual to the antisymmetric rank-two tensor

$$\hat{F}_{ab} = \frac{1}{6} \varepsilon_{abcde} F_{cde} = \varepsilon_{abcde} \, \eta_c \frac{\partial}{\partial \eta_d} A_e \quad . \tag{23}$$

The usual dual tensor $\hat{f}_{\mu\nu}$ in Euclidean space is related to $\hat{F}_{ab}$ by

$$\kappa^{-2} \hat{f}_{\mu\nu} = \hat{F}_{\mu\nu} - x_\mu \hat{F}_{5\nu} - x_\nu \hat{F}_{\mu 5} \quad . \tag{24}$$

In terms of the tensors F_{abc} and $\hat{F}_{ab}$ the Maxwell equations become

$$L_{ab} F_{abc} = 2e \, J_c \quad , \tag{25a}$$

$$L_{ab}\hat{F}_{bc} = \hat{F}_{ac} \quad , \tag{25b}$$

with J_c the electromagnetic current, which satisfies the conservation equation[9]

$$L_{ab}J_b = J_a \tag{26}$$

An explicit demonstration that Eqs. (25) and (26) indeed do correspond to the Maxwell and current conservation equations in x-space will be given below in Sec. 2(d).

When Eqs. (22) and (23) are used to express the Maxwell equations in terms of the potential A, Eq. (25b) is trivially satisfied, while Eq. (25a) becomes

$$P_{ca}A_a = 2e\, J_c \quad , \tag{27}$$

with P_{ca} the wave operator

$$P_{ca} = 2\, L_{cb}L_{ba} - 6\, L_{ca} + L^2\, \delta_{ca} \quad . \tag{28}$$

Using the angular momentum commutation relations it is straightforward to verify that P_{ca} has the following properties,

$$\begin{aligned} L_{bc}P_{ca} &= P_{bc}L_{ca} = P_{ba} \quad , \\ \eta_b P_{ba} &= P_{ba}\eta_a = 0 \quad , \end{aligned} \tag{29}$$

which guarantee the consistency of Eq. (27) with the constraints on J given by Eq. (7) and Eq. (26). Eq. (27) can be further simplified if the potential A_a is chosen to satisfy the condition

$$L_{ab}A_b = A_a \quad , \tag{30}$$

which is the hyperspherical analog of the Lorentz condition. When acting on potentials which obey Eq. (30) the operator P_{ca} becomes simply

$(L^2-4)\,\delta_{ca}$. Hence the wave equation becomes

$$(L^2-4)A_c = 2e\,J_c \quad , \tag{31}$$

and as expected involves the same wave operator as appears in the photon propagator equation, Eq. (11).

2(c) Electron Propagator Equation and Field Equation

To write the electron propagator equation we introduce the matrix γ_{ab} defined by

$$\gamma_{ab} = \frac{i}{4}\,[\alpha_a, \alpha_b] \quad . \tag{32}$$

Using the abbreviation $\gamma_{ab}L_{ab} = \gamma \cdot L$, we find that the electron propagator obeys the wave equation

$$(i\overrightarrow{\gamma \cdot L_1}+2)\,\frac{(\alpha\cdot\eta_1-1)(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4} = -\,2\pi^2\delta_S(\eta_1-\eta_2)(\alpha\cdot\eta_2+1) \tag{33}$$

$$\frac{(\alpha\cdot\eta_1-1)(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4}\,(i\overleftarrow{\gamma\cdot L_2}-2) = -\,2\pi^2\delta_S(\eta_1-\eta_2)(\alpha\cdot\eta_1-1) \quad ,$$

where the coefficient of the δ_S-function in Eq. (33) is obtained by averaging over the hypersphere, as in Eq. (13). An alternative method for obtaining Eq. (33) is to use the following relation between the electron and photon propagators,

$$\frac{(\alpha\cdot\eta_1-1)(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4} = -\,\tfrac{1}{2}(i\overrightarrow{\gamma\cdot L_1}+1)\,\frac{1}{(\eta_1-\eta_2)^2}\,(\alpha\cdot\eta_2+1)\ . \tag{34}$$

Applying the wave operator $i\overrightarrow{\gamma\cdot L_1}+2$ to Eq. (34) and using the identity

$$(i\gamma\cdot L_1+2)(i\gamma\cdot L_1+1) = -\,\tfrac{1}{2}(L_1{}^2-4) \quad , \tag{35}$$

we find

$$(i\overrightarrow{\gamma\cdot L_1}+2)\,\frac{(\alpha\cdot\eta_1-1)(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4} = \frac{1}{4}(L_1^{\,2}-4)\,\frac{1}{(\eta_1-\eta_2)^2}(\alpha\cdot\eta_2+1)$$

$$= -2\pi^2\,\delta_S(\eta_1-\eta_2)(\alpha\cdot\eta_2+1) \quad , \tag{36}$$

where in the last step we have used the photon propagator equation, Eq. (11).

The matrix γ_{ab} is a generalized spin operator for the electron. Writing

$$S_{ab} = -i\gamma_{ab} = \frac{1}{4}[\alpha_a, \alpha_b] \quad , \tag{37}$$

we find that S and L satisfy identical commutation relations,

$$\begin{aligned}[S_{ab}, S_{cd}] &= \delta_{ac}S_{db} - \delta_{ad}S_{cb} + \delta_{bc}S_{ad} - \delta_{bd}S_{ac} \quad , \\ [L_{ab}, L_{cd}] &= \delta_{ac}L_{db} - \delta_{ad}L_{cb} + \delta_{bc}L_{ad} - \delta_{bc}L_{ac} \quad ,\end{aligned} \tag{38}$$

and that

$$\begin{aligned}S^2 &= -5 \quad , \\ (L\cdot S)^2 &= 3\,L\cdot S - \tfrac{1}{2}L^2 \quad .\end{aligned} \tag{39}$$

The second relation in Eq. (39) leads immediately to the identity in Eq. (35).

Finally, in terms of an 8-component electron spinor χ the electron wave equation takes the form

$$\left\{ i\gamma_{ab}\left[\eta_a\left(\frac{\partial}{\partial\eta_b} - ieA_b(\eta)\right) - \eta_b\left(\frac{\partial}{\partial\eta_a} - ieA_a(\eta)\right)\right] + 2\right\}\chi = 0 \quad , \tag{40}$$

with the adjoint equation

$$\begin{aligned}\bar{\chi}\left\{ i\gamma_{ab}\left[\eta_a\left(\frac{\overleftarrow{\partial}}{\partial\eta_b} + ieA_b(\eta)\right) - \eta_b\left(\frac{\overleftarrow{\partial}}{\partial\eta_a} + ieA_a(\eta)\right)\right] - 2\right\} &= 0 \quad , \\ \bar{\chi} = \chi^+ \quad .&\end{aligned} \tag{41}$$

The electromagnetic current J_c which appears in Eq. (31) is given by

$$J_c(\eta) = \frac{-i}{2}\, \bar{\chi}\, [\alpha\cdot\eta, \alpha_c]\, \chi \quad . \tag{42}$$

Using the relation

$$\left(\eta_a \frac{\partial}{\partial\eta_c} - \eta_c \frac{\partial}{\partial\eta_a}\right) [\alpha\cdot\eta, \alpha_c] = [\alpha\cdot\eta, \alpha_a] - 2i\, \eta_a \gamma\cdot L \tag{43}$$

and Eqs. (40) and (41), we see that the current J_c satisfies the current conservation condition of Eq. (26). The constraint imposed by Eq. (7) is also obviously satisfied.

2(d) Transformation from the Hypersphere to x-Space

We give in this section the explicit transformations which map the hyperspherical Feynman rules and equations of motion into the corresponding rules and equations of motion in x-space. We begin with the Feynman rules of Eq. (9) and consider first a closed fermion loop coupling to $2n$ photons, given by

$$\frac{1}{2n} \sum_{\substack{\text{permutations}\\ \text{of } 1,\ldots,2n}} - \mathrm{tr}_8 \left\{ \frac{-i}{\pi^2} \frac{\frac{1}{2}(\alpha\cdot\eta_1 - 1)\frac{1}{2}(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4}\, ie\alpha_{a_2} \frac{-i}{\pi^2} \frac{\frac{1}{2}(\alpha\cdot\eta_2-1)\frac{1}{2}(\alpha\cdot\eta_3+1)}{(\eta_2-\eta_3)^4} \right.$$
$$\left. \cdots \frac{-i}{\pi^2} \frac{\frac{1}{2}(\alpha\cdot\eta_{2n}-1)\frac{1}{2}(\alpha\cdot\eta_1+1)}{(\eta_{2n}-\eta_1)^4}\, ie\alpha_{a_1} \right\} . \tag{44}$$

In order to obtain the corresponding 2n-point function in x-space, we must transform each of the $2n$ current indices according to the recipe of Eq. (8a), which means that we effectively make the replacement

$$\alpha_{a_i} \to \kappa_i^{\,3} \left(\alpha_{\mu_i} - x_{\mu_i} \alpha_5\right) \tag{45}$$

for each vertex α_{a_i}. After this replacement has been made, we must then find that a purely algebraic rearrangement of factors gives the 2n-point function computed from x-space Feynman rules. For the denominator in Eq. (44) the rearrangement is trivial, since substitution of Eq. (6) shows that

$$(\eta_i-\eta_{i+1})^2 = \kappa_i\kappa_{i+1}(x_i-x_{i+1})^2 \quad . \tag{46}$$

To rearrange the numerator we exploit the fact that the factors $\alpha\cdot\eta \pm 1$ appearing in each propagator are projection operators, allowing us to rewrite the numerator of the general propagator according to

$$\tfrac{1}{2}(\alpha\cdot\eta_i-1)\ \tfrac{1}{2}(\alpha\cdot\eta_{i+1}+1) = \tfrac{1}{2}(\alpha\cdot\eta_i-1)\ \tfrac{1}{2}\alpha\cdot(\eta_{i+1}-\eta_i)\ \tfrac{1}{2}(\alpha\cdot\eta_{i+1}+1) \quad . \tag{47}$$

We next introduce the matrix $O(x)$ given by

$$O(x) = \frac{1+\alpha_5\alpha\cdot x}{(1+x^2)^{\frac{1}{2}}} \quad , \qquad O(x)^{-1} = \frac{1-\alpha_5\alpha\cdot x}{(1+x^2)^{\frac{1}{2}}} \quad , \tag{48}$$

where $\alpha\cdot x$ denotes the 4-dimensional scalar product $\alpha_\mu x_\mu$. Some straightforward algebra then shows that

$$\begin{aligned} &O(x_i)\ \tfrac{1}{2}\alpha\cdot(\eta_i-\eta_{i+1})\ O(x_{i+1})^{-1} \\ &\qquad = \frac{\alpha\cdot(x_i-x_{i+1})}{[(1+x_i^2)(1+x_{i+1}^2)]^{\frac{1}{2}}} = \left(\frac{\kappa_i}{2}\ \frac{\kappa_{i+1}}{2}\right)^{\frac{1}{2}} \alpha\cdot(x_i-x_{i+1}) \end{aligned} \tag{49}$$

and that

$$O(x)\tfrac{1}{2}(1+\alpha\cdot\eta)(\alpha_\mu-x_\mu\alpha_5)\tfrac{1}{2}(\alpha\cdot\eta-1)O(x)^{-1} = \tfrac{1}{2}(1+\alpha_5)\alpha_\mu\tfrac{1}{2}(\alpha_5-1) \quad . \tag{50}$$

Substituting Eq. (4) for the α-matrices, we can pull all factors $\tfrac{1}{2}(\alpha_5\pm1) = \tfrac{1}{2}(\tau_3\pm1)$ to the left, where they combine to give a single factor $\tfrac{1}{2}(\tau_3+1)$. The factors τ_1 appearing in the matrices α_μ then cancel in pairs $(\tau_1^2=1)$, leaving

$$-\mathrm{tr}_8[\tfrac{1}{2}(\tau_3+1)\ X\{\gamma\}] = -\ \mathrm{tr}_4[X\{\gamma\}] \quad , \tag{51}$$

where $X\{\gamma\}$ contains γ-matrices only. The factor κ_i^3 appearing in Eq.

(45) precisely cancels the factor $(\kappa_i^{\frac{1}{2}}/\kappa_i^2)^2$ arising from the substitution of Eq. (49) and Eq. (46) into Eq. (44). Thus, we have shown that when the replacements of Eq. (45) are made, Eq. (44) can be algebraically rearranged to the form

$$\frac{1}{2n} \sum_{\substack{\text{permutations} \\ \text{of } 1,\ldots,2n}} - \mathrm{tr}_4 \left\{ \frac{-i}{2\pi^2} \frac{\gamma\cdot(x_1-x_2)}{(x_1-x_2)^4} ie\gamma_{\mu_2} \frac{-i}{2\pi^2} \frac{\gamma\cdot(x_2-x_3)}{(x_2-x_3)^4} \cdots \frac{-i}{2\pi^2} \frac{\gamma\cdot(x_{2n}-x_1)}{(x_{2n}-x_1)^4} ie\gamma_{\mu_1} \right\} , \tag{52}$$

which is just the 2n-point function calculated according to the Euclidean x-space Feynman rules.

The next step is to verify that the hyperspherical rule,

$$\int d\Omega_{\eta_1} d\Omega_{\eta_2} J_{a_1}(\eta_1) J_{a_2}(\eta_2) \frac{1}{4\pi^2} \frac{\delta_{a_1 a_2}}{(\eta_1-\eta_2)^2} , \tag{53}$$

correctly describes the propagation of a virtual photon from η_1 to η_2. The use of the current J in Eq. (53) is of course just a convenient shorthand for describing the 2n-point functions from which the photon is emitted (absorbed), with all variables other than those referring to the virtual photon in question suppressed. Calculating the Jacobian of the transformation of Eq. (6) by use of 5-dimensional spherical coordinates gives

$$d\Omega_\eta = \kappa^4 d^4x . \tag{54}$$

Substituting Eq. (54) into Eq. (53), using Eq. (46) to rewrite the denominator of the photon propagator and Eq. (8b) to reexpress the current J_a in terms of the combination of components $j_\mu = \kappa^3(J_\mu - x_\mu J_5)$ which is relevant to x-space, we find that the factors κ precisely cancel, leaving

$$\int d^4x_1 d^4x_2\, j_{\mu_1}(x_1) j_{\mu_2}(x_2) \frac{1}{4\pi^2} \Delta_{\mu_1\mu_2}(x_1, x_2) \quad , \tag{55}$$

with

$$\Delta_{\mu_1\mu_2}(x_1, x_2) = \frac{1}{(x_1-x_2)^2}\left[\delta_{\mu_1\mu_2} - \frac{2(x_1)_{\mu_1}(x_1)_{\mu_2}}{1+x_1^2} - \frac{2(x_2)_{\mu_1}(x_2)_{\mu_2}}{1+x_2^2} + \frac{4(1+x_1\cdot x_2)(x_1)_{\mu_1}(x_2)_{\mu_2}}{(1+x_1^2)(1+x_2^2)}\right] \tag{56a}$$

$$= \frac{\delta_{\mu_1\mu_2}}{(x_1-x_2)^2} + \frac{\partial}{\partial(x_2)_{\mu_2}}\left[\ell n(x_1-x_2)^2 \tfrac{1}{2}\frac{\partial}{\partial(x_1)_{\mu_1}} \ell n(1+x_1^2)\right] + \frac{\partial}{\partial(x_1)_{\mu_1}}\left[\ell n(x_1-x_2)^2 \tfrac{1}{2}\frac{\partial}{\partial(x_2)_{\mu_2}} \ell n(1+x_2^2)\right] - \frac{\partial}{\partial(x_1)_{\mu_1}}\frac{\partial}{\partial(x_2)_{\mu_2}}\left[\tfrac{1}{2}\ell n(1+x_1^2)\ell n(1+x_2^2)\right] . \tag{56b}$$

In Eq. (56a) we give the form of the x-space photon propagator $\Delta_{\mu_1\mu_2}(x_1, x_2)$ which emerges directly from the substitution of Eq. (8b) into Eq. (53); in Eq. (56b) we show that Δ can be rewritten as the usual Feynman propagator plus total derivative terms (gauge terms), which make no contribution to Eq. (55) because of electromagnetic current conservation. Hence Eq. (53) is completely equivalent to the usual x-space Feynman rules for propagating a virtual photon. In the next section we will show that the special significance of the gauge terms in Eq. (56a) is that they give Δ simple transformation properties under coordinate inversion.

To transform the photon and electron equations of motion and the current conservation equation to x-space, we rewrite the differential operators $\partial/\partial\eta_a$ and L_{ab} in terms of x-derivatives according to

$$\frac{\partial}{\partial\eta_a} = \kappa^{-1}(\delta_{a\mu} - \delta_{a5}x_\mu)\frac{\partial}{\partial x_\mu} + \text{terms proportional to } \eta_a \quad , \tag{57}$$

$$L_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \quad , \tag{58a}$$

$$L_{5\mu} = x_\mu\, x \cdot \frac{\partial}{\partial x} + (1-\kappa^{-1}) \frac{\partial}{\partial x_\mu} \quad ,$$

and use the following equation [obtained from Eq. (6)] to differentiate κ,

$$\frac{\partial \kappa}{\partial x_\mu} = -\kappa^2 x_\mu \quad . \tag{58b}$$

Applying Eq. (58) to Eqs. (21)-(24) we find that Eq. (24) implies the usual connection between the x-space field strength $\hat{f}_{\mu\nu}$ and potential a_μ,

$$f_{\lambda\sigma} \equiv \frac{\partial}{\partial x_\lambda} a_\sigma - \frac{\partial}{\partial x_\sigma} a_\lambda = \tfrac{1}{2} \varepsilon_{\lambda\sigma\mu\nu} \hat{f}_{\mu\nu} \quad . \tag{59}$$

Similarly, applying Eq. (58) to Eqs. (25) and (26), we find (after considerable algebra) that these equations reduce to the usual Maxwell and current conservation equations in x-space,

Eq. (25a) $\Longrightarrow$ $\partial_\mu f_{\mu\nu} = e\, j_\nu$, (60a)

Eq. (25b) $\Longrightarrow$ $\partial_\mu \hat{f}_{\mu\nu} = 0$, (60b)

Eq. (26) $\Longrightarrow$ $\partial_\mu j_\mu = 0$. (61)

To transform the electron wave equation [Eq. (40)] and the expression for the electromagnetic current [Eq. (42)] to x-space, we first note that the Pauli matrix τ_2 commutes with the wave operator in Eq. (40), and hence the 4-component spinors

$$\chi_\pm = \tfrac{1}{2}(1 \pm \tau_2)\chi \tag{62}$$

also satisfy Eq. (40). Defining x-space 4-component spinors $\psi_\pm$ by

$$\psi_\pm = \kappa^{3/2}\, O(x)\, \chi_\pm \quad , \tag{63}$$

we find that the projection of Eq. (42) into x-space takes the form

$$j_\mu(x) = \kappa^3 \frac{-i}{2} \quad \bar{\chi}\,[\alpha\cdot\eta, \alpha_\mu - x_\mu\alpha_5]\,\chi$$
$$= \quad j_\mu^+ - j_\mu^- \qquad , \tag{64}$$

with

$$j_\mu^\pm = \bar{\psi}_\pm \gamma_\mu \psi_\pm \quad , \quad \bar{\psi}_\pm = \psi^\dagger_\pm \quad . \tag{65}$$

Since a direct (and again somewhat lengthy) calculation shows that the Dirac wave operator obeys the transformation

$$\kappa^{3/2} O(x)\left\{ i\gamma_{ab}\left[\eta_a\left(\frac{\partial}{\partial\eta_b} - ieA_b(\eta)\right) - \eta_b\left(\frac{\partial}{\partial\eta_a} - ieA_a(\eta)\right)\right] + 2\right\} \kappa^{-3/2} O(x)^{-1}$$
$$= - i\tau_2\kappa^{-1}\gamma\cdot(\frac{\partial}{\partial x} - iea) \qquad , \tag{66}$$

the x-space spinors $\psi_\pm$ satisfy the usual mass-zero Dirac equation

$$\gamma\cdot(\frac{\partial}{\partial x} - iea)\psi_\pm = 0 \qquad . \tag{67}$$

This completes the demonstration that the hyperspherical formalism is completely equivalent to the usual formulation of quantum electrodynamics in Euclidean x-space.

2(e) Interpretation of Symmetries on the Hypersphere

We briefly discuss in this section the x-space interpretation of the rotational and inversion symmetries on the hypersphere. As we have seen, the photon wave operator is $L^2 - 4$, and this commutes with the ten generators L_{ab} of rotations on the hypersphere. Similarly, the free Dirac wave operator $i\gamma\cdot L + 2$ commutes with the ten operators

$$J_{ab} = L_{ab} + S_{ab} \qquad , \tag{68}$$

which are the hyperspherical rotation generators when spin is taken into account. We can interpret the hyperspherical rotational symmetry as follows. Six of the rotational generators $L_{\mu\nu}$ (or $J_{\mu\nu}$) leave the 5-axis invariant, and therefore, by Eq. (6b), leave x^2 unchanged. These clearly correspond in x-space to the generators of the homogeneous Lorentz group [which of course, in the Euclidean metric which we use, has become the four-dimensional rotation group O(4)]. The remaining four generators $L_{5\nu}$, which change x^2, correspond to rather complicated conformal transformations in x-space. For example, the 5-dimensional rotation

$$\begin{aligned} \eta \to \eta': \quad \eta'_{1,2,3} &= \eta_{1,2,3} \\ \eta'_4 &= \eta_4 \cos\alpha - \eta_5 \sin\alpha \\ \eta'_5 &= \eta_4 \sin\alpha + \eta_5 \cos\alpha \end{aligned} \tag{69a}$$

corresponds in x-space to the conformal transformation

$$\begin{aligned} x \to x': \quad x'_{1,2,3} = x_{1,2,3}/D\,, \quad x'_4 &= [x_4 \cos\alpha - \tfrac{1}{2}(1-x^2)\sin\alpha]/D\,, \\ D &= 1 + x^2 \sin^2\frac{\alpha}{2} + x_4 \sin\alpha \quad . \end{aligned} \tag{69b}$$

From this point of view, the manifest covariance of the hyperspherical Feynman rules under rotations generated by $L_{5\nu}$ is a reflection of the conformal invariance of zero-fermion-mass electrodynamics. We note, finally, that the ordinary x-space translation $x \to x' = x + a$ does not correspond to a linear transformation on η, but rather to the nonlinear transformation

$$\begin{aligned} \eta \to \eta': \quad \eta'_\mu &= [\eta_\mu + (1+\eta_5)a_\mu]/D' \quad , \\ \eta'_5 &= [\eta_5 - \tfrac{1}{2} a^2(1+\eta_5) - \eta\cdot a]/D' \quad , \\ D' = 1 + \tfrac{1}{2} a^2(1+\eta_5) &+ \eta\cdot a\,, \quad \eta\cdot a = \eta_\mu a_\mu \quad . \end{aligned} \tag{70}$$

Translation invariance of the x-space formalism guarantees that the 5-

dimensional formalism is covariant under the conformal transformations of Eq. (70), even though this is not manifestly evident.

In addition to the continuous-parameter rotation group, there is an important discrete symmetry operation on the hypersphere, the inversion

$$\eta \rightarrow -\eta \quad . \tag{71a}$$

According to Eq. (6), this corresponds in x-space to the inverse radius transformation

$$x \rightarrow -x/x^2 \quad . \tag{71b}$$

Because the trace of an odd number of α-matrices vanishes, the hyperspherical expression for the closed loop 2n-point function in Eq. (44) is invariant under simultaneous inversion of all the coordinates $\eta_1 \ldots \eta_{2n}$. Similarly, the hyperspherical photon propagator is inversion invariant. Hence we conclude that (as long as no divergent vacuum polarization insertions are made) the radiative corrected 2n-point functions in the hyperspherical formalism are manifestly inversion invariant. This in turn implies simple transformation properties for the corresponding x-space 2n-point function under simultaneous inverse radius transformation of the coordinates $x_1, \ldots, x_{2n}$. To find the form of the x-space transformation, we follow the notation of Eq. (53), and let $J_a(\eta)$ describe the emission of a photon from the 2n-point function at coordinate η, with the other 2n - 1 variables suppressed. In this notation, the inversion invariance of the 2n-point function reads

$$J_a(\eta' = -\eta) = J_a(\eta) \quad , \tag{72}$$

where, of course, the suppressed variables are also inverted. Projecting

$J_a(\eta)$ back to the x-space gives

$$\kappa^{-3} j_\mu(x) = J_\mu(\eta) - x_\mu J_5(\eta) \quad , \tag{73}$$

while projecting the inverted current $J_a(\eta')$ gives

$$(\kappa')^{-3} j_\mu(x') = J_\mu(\eta') - x'_\mu J_5(\eta') \quad , \tag{74}$$

with

$$\eta' = -\eta \ , \ x' = 1 + \eta'_5 = 1 - \eta_5 \ , \ x'_\mu = -x_\mu / x^2 \ . \tag{75}$$

Using Eqs. (73) and (74), we can convert the equality of Eq. (72) into a relation between $j_\mu(x')$ and $j_\mu(x)$, giving

$$j_\mu(x) = (x^2)^{-3} M_{\mu\nu}(x) j_\nu(x') \quad , \tag{76}$$

where

$$M_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \quad ; \quad M_{\mu\nu}(x) M_{\nu\sigma}(x) = \delta_{\mu\sigma} \quad . \tag{77}$$

Thus, the 2n-point function in x-space is left invariant under the combined operations of (i) simultaneous inverse radius transformation $x_j \rightarrow -x_j/x_j^2$, $j = 1, \ldots, 2n$ and (ii) application of the projection operator $\prod_{j=1}^{2n} (x_j^2)^{-3} \times \prod_{j=1}^{2n} M_{\mu'_j \mu_j}(x_j)$ to the vector indices. This recipe is just the one discussed by Schreier.[2] In terms of the matrix $M_{\mu\nu}$ we can understand the significance of the gauge terms in the x-space photon propagator of Eq. (56): the gauge terms guarantee that under inverse radius transformations the photon propagator transforms covariantly, i.e.

$$\Delta_{\mu_1\mu_2}(x_1, x_2) = M_{\mu_1\mu'_1}(x_1) M_{\mu_2\mu'_2}(x_2) \Delta_{\mu'_1\mu'_2}(-x_1/x_1^2, -x_2/x_2^2) \quad . \tag{78}$$

The usual Feynman propagator, of course, does not satisfy Eq. (78).

3. CONNECTION WITH THE MANIFESTLY CONFORMAL-COVARIANT FORMALISM

In this section we discuss massless, Euclidean electrodynamics in the manifestly conformal-covariant O(5,1) language, and develop its connection with the 5-dimensional formalism of the preceding section. In Sec. 3(a) we review the O(5,1) formalism and in Sec. 3(b) we develop, in a heuristic fashion, the O(5,1) Feynman rules for electrodynamics. In Sec. 3(c) we show that the O(5,1) rules are related to the 5-dimensional rules by a simple projective transformation.

3(a) The O(5,1) Formalism

As has been greatly emphasized recently,[1] a large class of renormalizable field theories containing no dimensional parameters (masses or dimensional coupling constants) are invariant under the 15-parameter conformal group of transformations on space-time. In particular, quantum electrodynamics with zero fermion mass is conformal invariant. We recall that of the 15 conformal-group generators, 10 are the generators of the Poincaré group, 1 generates the dilatations

$$x_\mu \rightarrow \lambda x_\mu \tag{79}$$

and the remaining 4 generate the special conformal transformations

$$x_\mu \rightarrow (x_\mu + c_\mu x^2)/(1+2c\cdot x+c^2x^2) \quad . \tag{80}$$

Although the usual formulations of massless field theories are manifestly Poincaré invariant, their invariance under the nonlinear transformations of Eq. (80) is not manifestly evident. However, a very pretty way of achieving manifest conformal invariance was introduced by Dirac,[10] and has been further developed recently. The basic idea is to replace the usual field equations over the Minkowski space-time manifold x_μ by equivalent field equations over a 6-dimensional projective manifold ξ_A. [We adopt the convention that 6-dimensional vector indices are indicated by capital Roman letters A, B, ... which take the values 1, ..., 6.] The coordinate x is related to ξ_A by the projective transformation

$$x = \xi_\mu / \xi_+ \ , \ \ \xi_+ = \xi_5 + \xi_6 \ . \tag{81}$$

When the metric in x-space is the Minkowski metric (1, 1, 1, -1), the ξ-space is endowed with the metric (1, 1, 1, -1, 1, -1); correspondingly, when the metric in x-space is the Euclidean metric (1, 1, 1, 1), the ξ-space is endowed with the metric (1, 1, 1, 1, 1, -1). In either case, if ξ is restricted to the light-cone

$$\xi^2 = 0 \quad , \tag{82}$$

then it can be shown that the 15-parameter linear group of pseudorotations on ξ is isomorphic to the conformal group of nonlinear transformations on x. In the Minkowski case, the pseudorotations form the pseudoorthogonal group O(4, 2), while in the Euclidean case with which we are primarily concerned, they form the pseudoorthogonal group O(5, 1). So to construct a *manifestly conformal invariant formulation* of massless, Euclidean electrodynamics, we must write equations which are manifestly covariant under the operations of O(5, 1).

Because excellent reviews are available in the literature,[11] we will not actually detail the development of the O(5, 1)-covariant formalism, but rather will simply summarize the results needed for the construction of Feynman rules.

(1) The electromagnetic current is represented by a 6-vector $J_A(\xi)$, homogeneous in ξ of degree -3 and satisfying the kinematic constraint

$$\xi \cdot J(\xi) = 0 \quad . \tag{83}$$

The equation of electromagnetic current conservation takes the form[9]

$$L_{AB} J^B(\xi) = J_A(\xi) \quad , \tag{84a}$$

with

$$L_{AB} = \xi_A \frac{\partial}{\partial \xi^B} - \xi_B \frac{\partial}{\partial \xi^A} \quad , \tag{84b}$$

and $J_A(\xi)$ is related to the x-space current $j_\mu(x)$ by the recipe

$$j_\mu(x) = \xi_+{}^3 \{ J_\mu(\xi) - x_\mu [J_5(\xi) + J_6(\xi)] \} \quad . \tag{85}$$

Note that Eqs. (84) and (85) are both invariant under "gauge" transformations of the form

$$J_A(\xi) \rightarrow J_A(\xi) + \xi_A M(\xi) \quad , \tag{86}$$

with $M(\xi)$ homogeneous in ξ of degree -4. The invariance of Eq. (85) follows immediately from Eq. (81), while Eq. (84a) is left unchanged because

$$L_{AB}\, \xi^B M(\xi) = \xi_A (5 + \xi \cdot \frac{\partial}{\partial \xi}) M(\xi) = \xi_A M(\xi) \quad , \tag{87}$$

where in the second equality we have used the homogeneity of $M(\xi)$.

(2) The electromagnetic potential is represented by a 6-vector $A_B(\xi)$, homogeneous in ξ of degree -1 and satisfying the constraint $\xi \cdot A = 0$. The photon wave equation takes the form

$$\Box_6 A_B(\xi) = e\, J_B(\xi) \quad , \tag{88a}$$

with

$$\Box_6 = \frac{\partial}{\partial\xi_B}\,\frac{\partial}{\partial\xi^B} \quad . \tag{88b}$$

(3) The electron field is represented by an 8-component spinor $\chi(\xi)$, homogeneous in ξ of degree -2, which obeys the wave equation

$$\left\{i\gamma_{AB}\left[\xi^A\left(\frac{\partial}{\partial\xi_B} - ieA^B(\xi)\right) - \xi^B\left(\frac{\partial}{\partial\xi_A} - ieA^A(\xi)\right)\right] + 2\right\}\chi = 0 \, . \tag{89}$$

The matrix γ_{AB} is defined by

$$\gamma_{AB} = \frac{i}{4}\,[\beta_A, \beta_B] \quad , \tag{90}$$

where the 8x8 matrices β_A satisfy a Clifford algebra

$$\{\beta_A, \beta_B\} = 2\, g_{AB} \tag{91}$$

with g_{AB} the metric tensor. An explicit representation of the β's is

$$\beta_\mu = -\gamma_\mu \tau_3 \qquad \beta_5 = \tau_1 \qquad \beta_6 = -i\tau_2 \quad . \tag{92}$$

The electromagnetic current of the electron is given, in terms of the spinor χ, by

$$J_A = 2\xi^B \bar{\chi}\,\gamma_{BA}\chi \quad , \quad \bar{\chi} = \chi^+\beta_6 \quad . \tag{93}$$

These equations completely specify the O(5,1)-covariant formulation of massless electrodynamics, and, via Eq. (85), allow us to project 2n-point functions in the 6-dimensional language back into 2n-point functions in x-space.

3(b) O(5,1)-Covariant Feynman Rules

We proceed next to deduce, in a heuristic fashion, Feynman rules for the O(5,1)-covariant calculation of closed-fermion-loop processes. We will not actually directly prove the equivalence of these rules with the usual x-space rules, but rather will show this indirectly in the next section by deducing the 5-dimensional rules of Sec. 2 from the 6-dimensional rules which we now develop. To begin, we infer from Eq. (93) that the rule for a vertex where a current with polarization index A acts at coordinate ξ is

$$\text{vertex} \propto e\Gamma_A(\xi) = e\,\xi^B[\beta_B, \beta_A] \quad ; \tag{94}$$

clearly, this rule automatically satisfies the kinematic constraint of Eq. (83). [In Eq. (94) and subsequent equations of the present section, we omit numerical proportionality constants.] Next, we must guess the rule for the electron propagator $S(\xi_1, \xi_2)$. We first note that since $\chi(\xi)$ is homogeneous in ξ of degree -2, S must be homogeneous of degree -2 in ξ_1 and ξ_2 independently. A check on this requirement is provided by the fact that since $J_A(\xi)$ is homogeneous of degree -3, a $2n$-point function must be homogeneous of degree -3 in each of the $2n$ coordinates. Since the vertex $\Gamma_A(\xi)$ is homogeneous of degree $+1$, this requirement will be satisfied by propagator-vertex chains of the form

$$S(\xi_1, \xi_2)\Gamma_{A_2}(\xi_2)S(\xi_2, \xi_3)\Gamma_{A_3}(\xi_3)\ldots \tag{95}$$

only if the propagator is homogeneous of degree -2 in each of its arguments. The homogeneity requirement immediately restricts the choice of propagator to one of two possible forms,

$$\begin{aligned} S_1(\xi_1, \xi_2) &= \frac{\beta\cdot\xi_1\,\beta\cdot\xi_2}{(\xi_1\cdot\xi_2)^3} \quad , \\ S_2(\xi_1, \xi_2) &= \frac{1}{(\xi_1\cdot\xi_2)^2} \quad . \end{aligned} \tag{96}$$

We can rule out S_1 as a possible choice, however, by noting that when

S_1 is sandwiched between the two adjacent vertices we get

$$\Gamma_{A_1}(\xi_1)S_1(\xi_1,\xi_2)\Gamma_{A_2}(\xi_2) = \frac{[\beta\cdot\xi_1,\beta_{A_1}]\beta\cdot\xi_1\beta\cdot\xi_2[\beta\cdot\xi_2,\beta_{A_2}]}{(\xi_1\cdot\xi_2)^3}$$

$$= -4\,\frac{\beta\cdot\xi_1(\xi_1)_{A_1}(\xi_2)_{A_2}\beta\cdot\xi_2}{(\xi_1\cdot\xi_2)^3}\ , \tag{97}$$

where we have used the fact that $(\beta\cdot\xi)^2 = \xi^2 = 0$. But as we have seen above, "gauge" terms of the form $(\xi_1)_{A_1}$ or $(\xi_2)_{A_2}$ project to a null current in x-space, so use of S_1 as the propagator would lead to identically vanishing 2n-point functions in x-space. We conclude that the correct choice of electron propagator is S_2, and that the O(5,1)-covariant expression for a closed fermion loop coupling to 2n-photons is given (up to proportionality constants) by

$$\sum_{\substack{\text{permutations}\\ \text{of } 1,\ldots,2n}} \operatorname{tr}_8\left\{\frac{1}{(\xi_1\cdot\xi_2)^2}[\beta\cdot\xi_2,\beta_{A_2}]\frac{1}{(\xi_2\cdot\xi_3)^2}\cdots\frac{1}{(\xi_{2n}\cdot\xi_1)^2}[\beta\cdot\xi_1,\beta_{A_1}]\right\}\ . \tag{98}$$

Although we have constructed our rules to satisfy the kinematic constraint of Eq. (83) and the requirements of homogeneity, we must now check whether they are consistent with the equation of current conservation, Eq. (84a). To do this, we first examine the effect of the free Dirac operator on the propagator forms S_1 and S_2. By direct calculation, we find that

$$(i\gamma_{AB}\overrightarrow{L_1^{AB}}+2)\,S_1(\xi_1,\xi_2) = S_1(\xi_1,\xi_2)(i\gamma_{AB}\overleftarrow{L_2^{AB}}-2) = 0\ , \tag{99a}$$

$$(i\gamma_{AB}\overrightarrow{L_1^{AB}}+2)\,S_2(\xi_1,\xi_2) = 2\,S_1(\xi_1,\xi_2)\ ,$$

$$S_2(\xi_1,\xi_2)(i\gamma_{AB}\overleftarrow{L_2^{AB}}-2) = -2\,S_1(\xi_1,\xi_2)\ , \tag{99b}$$

all for $\xi_1 \neq \xi_2$ [at $\xi_1 = \xi_2$ there are additional delta-function contributions, which we omit in writing Eq. (99)]. We see that the correct propagator S_2 does not satisfy the Dirac equation, and that adding in an arbitrary multiple of S_1 cannot fix things up. In effect, we see that S_1 is a null propagator (because it leads to a vanishing current in x-space) and that S_2 is a pseudo-propagator, which when acted on by the Dirac wave operator gives a multiple of the null propagator, but not zero.

Let us now examine the effect of this peculiar state of affairs on the current conservation properties of Eq. (98). We consider the propagator-vertex chain linking the points ξ_1, ξ, ξ_2 and act with the differential operator $L_{AB} = \xi_A\, \partial/\partial\xi^B - \xi_B\, \partial/\partial\xi^A$, giving

$$L_{AB} \cdots [\beta\cdot\xi_1, \beta_{A_1}] \frac{1}{(\xi_1\cdot\xi)^2} [\beta\cdot\xi, \beta^B] \frac{1}{(\xi\cdot\xi_2)^2} [\beta\cdot\xi_2, \beta_{A_2}] \cdots$$

$$= \ldots [\beta\cdot\xi_1, \beta_{A_1}] \frac{1}{(\xi_1\cdot\xi)^2} [\beta\cdot\xi, \beta_A] \frac{1}{(\xi\cdot\xi_2)^2} [\beta\cdot\xi_2, \beta_{A_2}] \cdots \qquad (100)$$

$$+ R_A \ .$$

The first term on the right-hand side of Eq. (100) is just the result required by Eq. (84a), while the remainder R_A is given by

$$R_A = -2\,\xi_A \Big\{ \ldots [\beta\cdot\xi_1, \beta_{A_1}] \frac{1}{(\xi_1\cdot\xi)^2} (i\gamma_{AB} \overrightarrow{L^{AB}} + 2) \frac{1}{(\xi\cdot\xi_2)^2} [\beta\cdot\xi_2, \beta_{A_2}] \cdots$$

$$+ \ldots [\beta\cdot\xi_1, \beta_{A_1}] \frac{1}{(\xi_1\cdot\xi)^2} (i\gamma_{AB} \overleftarrow{L^{AB}} - 2) \frac{1}{(\xi\cdot\xi_2)^2} [\beta\cdot\xi_2, \beta_{A_2}] \ldots \Big\} . \qquad (101)$$

Substituting Eq. (99b) and algebraically rearranging as in Eq. (97), we get[12]

$$R_A = 8\,\xi_A \Big\{ \dots [\beta\cdot\xi_1, \beta_{A_1}] \frac{1}{(\xi_1\cdot\xi)^2} \frac{\beta\cdot\xi(\xi_2)_{A_2}\beta\cdot\xi_2}{(\xi\cdot\xi_2)^3} \dots$$

$$+ \dots \frac{\beta\cdot\xi_1(\xi_1)_{A_1}\beta\cdot\xi}{(\xi_1\cdot\xi)^3} \frac{1}{(\xi\cdot\xi_2)^2} [\beta\cdot\xi_2, \beta_{A_2}] \dots \Big\} . \qquad (102)$$

Although Eq. (102) does not vanish, the first term in the curly bracket is a pure "gauge" term with respect to the index A_2, while the second is a pure "gauge" term with respect to the index A_1, and hence both give a vanishing contribution to the 2n-point function when projected back to x-space. So we see that because S_2 is a pseudo-propagator, Eq. (98) only satisfies a pseudo-current-conservation condition: when we test current conservation on a given index, Eq. (84a) is not satisfied in the 6-dimensional space, but does hold when we project on all of the remaining indices to transform back to x-space.

As an explicit illustration of this pseudo-conservation property, let us consider the single loop two-point function, which according to Eq. (98) is given by

$$\frac{\mathrm{tr}_8\{[\beta\cdot\xi_1, \beta_{A_1}]\,[\beta\cdot\xi_2, \beta_{A_2}]\}}{(\xi_1\cdot\xi_2)^4} \propto \frac{\xi_1\cdot\xi_2 g_{A_1A_2} - (\xi_1)_{A_2}(\xi_2)_{A_1}}{(\xi_1\cdot\xi_2)^4} , \qquad (103)$$

Acting on Eq. (84) with $(L_1)^{A_1'A_1}$ gives

$$(L_1)^{A_1'A_1} \frac{\xi_1\cdot\xi_2 g_{A_1A_2} - (\xi_1)_{A_2}(\xi_2)_{A_1}}{(\xi_1\cdot\xi_2)^4} = \frac{\xi_1\cdot\xi_2 g^{A_1'}{}_{A_2} - (\xi_1)_{A_2}(\xi_2)^{A_1'}}{(\xi_1\cdot\xi_2)^4} + R , \qquad (104)$$

$$R = -\frac{4\xi_1\cdot\xi_2\xi_1^{A_1'}(\xi_2)_{A_2}}{(\xi_1\cdot\xi_2)^5} .$$

As expected, there is an extra term R which, because it contains the factor $(\xi_2)_{A_2}$, makes no contribution to the two-point function in x-space. Interestingly, there is no way of modifying Eq. (103) to make the extra term R vanish. To see this we note that the only other second rank tensor with the correct homogeneity properties and which satisfies the kinematic constraint of Eq. (83) is

$$\frac{(\xi_1)_{A_1}(\xi_2)_{A_2}}{(\xi_1\cdot\xi_2)^4} \quad . \tag{105}$$

However, Eq. (87) tells us that this expression satisfies

$$(L_1)^{A_1' A_1} \frac{(\xi_1)_{A_1}(\xi_2)_{A_2}}{(\xi_1\cdot\xi_2)^4} = \frac{(\xi_1)^{A_1'}(\xi_2)_{A_2}}{(\xi_1\cdot\xi_2)^4} \quad , \tag{106}$$

so adding a multiple of Eq. (105) to Eq. (103) cannot cancel away R. We conclude that pseudo-current-conservation is an unavoidable feature of the O(5,1)-covariant formalism.

Next, we turn our attention to the photon propagator $D_{A_1A_2}(\xi_1, \xi_2)$. Because the photon field $A_B(\xi)$ is homogeneous in ξ of degree -1, the photon propagator D must be homogeneous of degree -1 in ξ_1 and ξ_2 independently. In addition, in order to annihilate the extra "gauge" terms which appear when we test current conservation on indices of the closed loop other than A_1, A_2, the photon propagator must be explicitly transverse,

$$\xi_1^{A_1} D_{A_1A_2}(\xi_1, \xi_2) = \xi_2^{A_2} D_{A_1A_2}(\xi_1, \xi_2) = 0 \quad . \tag{107}$$

The simplest form which satisfies these requirements is

$$D_{A_1A_2}(\xi_1,\xi_2) = \frac{1}{\xi_1\cdot\xi_2}\left[g_{A_1C} - \frac{(\tilde{\xi}_1)_{A_1}\xi_{1C}}{\tilde{\xi}_1\cdot\xi_1}\right]\left[g^C{}_{A_2} - \frac{\xi_2{}^C(\tilde{\xi}_2)_{A_2}}{\tilde{\xi}_2\cdot\xi_2}\right], \qquad (108)$$

where $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are arbitrary points which are held fixed when doing the virtual integrations over ξ_1 and ξ_2. Because of gauge invariance, closed-fermion-loop expressions have no dependence on $\tilde{\xi}_1$ and $\tilde{\xi}_2$ after one sums over all orderings, with respect to other photons which are present, of the emission and absorption of the virtual photon propagated by Eq. (108). The simplest way to verify this statement, and to check the correctness of Eq. (108) to begin with, is to transform Eq. (108) back to x-space. We find

$$\xi^4_{1+}\xi^4_{2+}J^{A_1}(\xi_1)D_{A_1A_2}(\xi_1,\xi_2)J^{A_2}(\xi_2) = -2j_{\mu_1}(x_1)\Delta'_{\mu_1\mu_2}(x_1,x_2)j_{\mu_2}(x_2) \;, \qquad (109)$$

with the effective x-space propagator given by

$$\Delta'_{\mu_1\mu_2}(x_1,x_2) = \frac{\delta_{\mu_1\mu_2}}{(x_1-x_2)^2} + 2\,\frac{(\tilde{x}_1-x_1)_{\mu_1}(x_1-x_2)_{\mu_2}}{(\tilde{x}_1-x_1)^2(x_1-x_2)^2} + 2\,\frac{(x_2-x_1)_{\mu_1}(\tilde{x}_2-x_2)_{\mu_2}}{(x_2-x_1)^2(\tilde{x}_2-x_2)^2}$$
$$-\,2\,\frac{(\tilde{x}_1-x_1)_{\mu_1}(\tilde{x}_2-x_2)_{\mu_2}}{(\tilde{x}_1-x_1)^2(\tilde{x}_2-x_2)^2} \qquad (110a)$$

$$= \frac{\delta_{\mu_1\mu_2}}{(x_1-x_2)^2} + \frac{\partial}{\partial(x_2)_{\mu_2}}\left[\ell n(x_1-x_2)^2\,\tfrac{1}{2}\,\frac{\partial}{\partial(x_1)_{\mu_1}}\,\ell n(\tilde{x}_1-x_1)^2\right]$$
$$+\frac{\partial}{\partial(x_1)_{\mu_1}}\left[\ell n(x_1-x_2)^2\,\tfrac{1}{2}\,\frac{\partial}{\partial(x_2)_{\mu_2}}\,\ell n(\tilde{x}_2-x_2)^2\right] - \frac{\partial}{\partial(x_1)_{\mu_1}}\,\frac{\partial}{\partial(x_2)_{\mu_2}}$$
$$\times\,[\tfrac{1}{2}\,\ell n(\tilde{x}_1-x_1)^2\,\ell n(\tilde{x}_2-x_2)^2]\,. \qquad (110b)$$

In Eq. (110a) we give the form of the x-space propagator which emerges directly from the transformation; in this equation $x_1, \tilde{x}_1, x_2, \tilde{x}_2$ denote, respectively, the x-space images of $\xi_1, \tilde{\xi}_1, \xi_2, \tilde{\xi}_2$. In Eq. (110b) we see that Δ' is equivalent, up to gauge terms, to the usual Feynman propagator, and in particular that all the dependence on $\tilde{x}_1, \tilde{x}_2$ is contained in the gauge terms. This verifies that Eq. (108) is a valid expression for the photon propagator, and that $\tilde{\xi}_1$ and $\tilde{\xi}_2$ drop out of gauge invariant quantities, such as closed fermion loops. [The derivation of Eq. (110) from the conformal-covariant expression of Eq. (108) indicates that the effect of the x-space gauge terms is to render Δ' covariant under x-space conformal transformations, provided that the points $\tilde{x}_1, \tilde{x}_2$ are conformally transformed along with x_1 and x_2.[13]]

Finally, calculation of the Jacobian of the transformation of Eq. (81) shows that

$$\int d^4x = \int dS_\xi \, \xi_+^{-4} \quad , \tag{111}$$

where $\int dS_\xi$ denotes an integration over the hypersphere $\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2 + \xi_4{}^2 + \xi_5{}^2 = \xi_6{}^2$, with ξ_6 held fixed. Comparing with Eq. (109), we see that

$$\begin{aligned} &\int d^4x_1 d^4x_2 (-2) j_{\mu_1}(x_1) \Delta'_{\mu_1\mu_2}(x_1, x_2) j_{\mu_2}(x_2) \\ &= \int dS_{\xi_1} dS_{\xi_2} \, J^{A_1}(\xi_1) D_{A_1A_2}(\xi_1, \xi_2) J^{A_2}(\xi_2) \quad , \end{aligned} \tag{112}$$

indicating that the Feynman rule for virtual integrations is simply

$$\text{virtual integration over } \xi: \quad \int dS_\xi \quad . \tag{113}$$

This completes our specification of the O(5,1)-covariant Feynman rules for calculating closed loop quantities.

3(c) Projection onto the 5-Dimensional Unit Hypersphere

We complete our discussion of the O(5,1)-covariant formalism by showing that it is related, by a simple projective transformation, to the 5-dimensional Feynman rules of Sec. 2. The transformation is generated by exploiting the fact that in an n-virtual photon process, the fixed points $\tilde{\xi}$ in each of the n photon propagators can be chosen independently, provided that over-all Bose symmetry is maintained. Since closed fermion loop amplitudes are independent of all of the propagator fixed points, they will be unchanged if we integrate all of the fixed points over their respective hyperspheres $\tilde{\xi}_1^{\,2}+\ldots+\tilde{\xi}_5^{\,2}=\tilde{\xi}_6^{\,2}$. The effect of this integration is to replace the photon propagator of Eq. (108) by the averaged propagator

$$\overline{D}_{A_1A_2}(\xi_1,\xi_2)=\frac{\int dS_{\tilde{\xi}_1}\,dS_{\tilde{\xi}_2}\,D_{A_1A_2}(\xi_1,\xi_2)}{\int dS_{\tilde{\xi}_1}\,dS_{\tilde{\xi}_2}}\quad . \tag{114}$$

The integrations in Eq. (114) are readily evaluated, giving

$$\overline{D}_{A_1A_2}(\xi_1,\xi_2)=\frac{1}{\xi_1\cdot\xi_2}\left[g_{A_1C}-\frac{g_{A_16}\,\xi_{1C}}{(\xi_1)_6}\right]\left[g^C{}_{A_2}-\xi_2{}^C\,\frac{g_{A_26}}{(\xi_2)_6}\right]$$
$$+\text{ terms proportional to }(\xi_1)_{A_1}\text{ or }(\xi_2)_{A_2}\quad ; \tag{115}$$

the terms proportional to $(\xi_1)_{A_1}$ or $(\xi_2)_{A_2}$ are uninteresting because they make a vanishing contribution by virtue of the constraint equation, Eq. (83). The key feature of Eq. (115) is that the quantities in brackets,

$$g_{A_1C}-\frac{g_{A_16}\,\xi_{1C}}{(\xi_1)_6}\quad ,\quad g^C{}_{A_2}-\xi_2{}^C\,\frac{g_{A_26}}{(\xi_2)_6} \tag{116}$$

both vanish when $C = 6$, so the sum in Eq. (115) extends only over

$C = 1, \ldots, 5$. This suggests projecting onto a 5-dimensional space, as follows: (i) The 5-dimensional coordinate η_a is related to the 6-dimensional coordinate ξ_A by

$$\eta_a = \frac{\xi_a}{\xi_6} \quad , \quad a = 1, \ldots, 5 \quad . \tag{117a}$$

The light-cone restriction on ξ implies that

$$\eta^2 = 1 \quad , \tag{117b}$$

and scalar products in 6-space may be rewritten in 5-space as follows,

$$\xi_1 \cdot \xi_2 = -\tfrac{1}{2}\, \xi_{16}\xi_{26}(\eta_1 - \eta_2)^2 \quad . \tag{117c}$$

(ii) The five-dimensional current $J_a(\eta)$ is related to the 6-dimensional current $J_A(\xi)$ by

$$J_a(\eta) = \xi_6^{\,3}[J_a(\xi) - \eta_a J_6(\xi)] \quad , \tag{118}$$

which is just the projection generated by the brackets of Eq. (116).

We proceed now to combine Eqs. (115), (117) and (118). Using

$$\int dS_\xi = \int d\Omega_\eta\, \xi_6^{\,4} \quad , \tag{119}$$

we get

$$\begin{aligned} &\int dS_{\xi_1} dS_{\xi_2} J^{A_1}(\xi_1)\, \overline{D}_{A_1A_2}(\xi_1, \xi_2) J^{A_2}(\xi_2) \\ &= \int d\Omega_{\eta_1} d\Omega_{\eta_2} J_{a_1}(\eta_1) \frac{-2\delta_{a_1a_2}}{(\eta_1 - \eta_2)^2} J_{a_2}(\eta_2) \quad , \end{aligned} \tag{120}$$

which reproduces the 5-dimensional Feynman rule for photon propagation. To study the effect of the projection operation of Eq. (118) on the O(5,1)-covariant expression for a closed fermion loop in Eq. (98), we consider first the projection of the vertex $\Gamma_A(\xi)$. We find

$$\xi_6{}^3[g_a{}^A - \eta_a g_6{}^A]\,\Gamma_A(\xi)$$

$$= \xi_6{}^3[\beta\cdot\xi, \beta_a - \eta_a\beta_6] = \xi_6{}^4[\beta\cdot\eta - \beta_6, \beta_a - \eta_a\beta_6] \tag{121}$$

$$= -\,\xi_6{}^4(\alpha\cdot\eta+1)\alpha_a(\alpha\cdot\eta-1) \quad ,$$

where we have introduced matrices α_a defined by

$$\alpha_a = -\,\beta_6\beta_a \quad . \tag{122}$$

Since the propagator $(\xi_1\cdot\xi_2)^{-2}$ can be rewritten as

$$\frac{1}{(\xi_1\cdot\xi_2)^2} = \frac{4}{(\xi_1)_6^2(\xi_2)_6^2(\eta_1-\eta_2)^4} \quad , \tag{123}$$

we see that the projection of Eq. (118) transforms Eq. (98) into

$$4^{2n} \sum_{\substack{\text{permutations}\\ \text{of } 1,\ldots,2n}} \mathrm{tr}_8 \left\{ \frac{1}{(\eta_1-\eta_2)^4}\,(\alpha\cdot\eta_2+1)\alpha_{a_2}(\alpha\cdot\eta_2-1)\,\frac{1}{(\eta_2-\eta_3)^4} \right.$$
$$\left. \cdots \frac{1}{(\eta_{2n}-\eta_1)^4}\,(\alpha\cdot\eta_{2n}+1)\alpha_{a_1}(\alpha\cdot\eta_1-1)\right\} \quad , \tag{124}$$

which apart from normalization constants is identical with Eq. (44). So we have verified that the projective transformation generated by using the averaged propagator of Eq. (114) just gives the 5-dimensional Feynman rules for the photon propagator, the electron propagator and the electron-photon vertex.

To conclude, we show that Eq. (118) and the formal properties of the 6-dimensional current $J_A(\xi)$ imply the corresponding formal properties of the 5-dimensional current $J_a(\eta)$. We begin with the constraint equation, $\xi\cdot J_A(\xi) = 0$, which can be rewritten as

$$0 = \xi_a J_a(\xi) - \xi_6 J_6(\xi) = \xi_6\eta_a[J_a(\xi) - \eta_a J_6(\xi)] = \xi_6^{-2}\,\eta\cdot J(\eta) \ , \tag{125}$$

giving the 5-dimensional constraint equation, Eq. (7). Next we consider the 6-dimensional version of current conservation,

$$L_{AB}\, J^{B}(\xi) = J_{A}(\xi) \quad , \tag{126}$$

and use Eq. (87), with $M(\xi) = \xi_6^{-1} J_6(\xi)$, to write

$$L_{AB}[J^{B}(\xi)-\xi^{B}\xi_6^{-1}J_6(\xi)] = J_{A}(\xi) - \xi_{A}\xi_6^{-1} J_6(\xi) \quad . \tag{127}$$

Since the sum on B in Eq. (127) exntends only over $B = 1,\ldots,5$, and since we are interested only in values of the free index $A = 1,\ldots,5$, no derivatives $\partial/\partial\xi_6$ appear. Hence, on multiplying through by $\xi_6{}^3$ and using the fact that

$$\xi_a \frac{\partial}{\partial\xi_b} - \xi_b \frac{\partial}{\partial\xi_a} = \eta_a \frac{\partial}{\partial\eta_b} - \eta_b \frac{\partial}{\partial\eta_a} = L_{ab} \quad , \tag{128}$$

Eq. (127) becomes the 5-dimensional current conservation equation

$$L_{ab}\, J_b(\eta) = J_a(\eta) \quad . \tag{129}$$

4. DISCUSSION

In this section we very briefly discuss possible generalizations and applications of the 5-dimensional formalism of Sec. 2. First, we note that although we have worked with a Euclidean x-space metric throughout, it should be straightforward to generalize the 5-dimensional rules to the usual Minkowski case. The hypersphere will then become the hyperbolic domain

$$\eta_1{}^2 + \eta_2{}^2 + \eta_3{}^2 - \eta_4{}^2 + \eta_5{}^2 = 1 \quad , \tag{130}$$

which is a (4,1) de Sitter space of unit radius.[14] A further generlization would consist of giving the electron a mass m and, since the distance scale now acquires a meaning, calling the radius of the de Sitter space R, so that Eq. (130) becomes

$$\eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_4^2 + \eta_5^2 = R^2 \quad . \tag{131}$$

As Dirac[5] has shown, an appropriate wave equation describing a massive electron in a de Sitter space of radius R is

$$\left\{ i\gamma_{ab}\left[\eta_a\left(\frac{\partial}{\partial\eta_b} - ieA_b(\eta)\right) - \eta_b\left(\frac{\partial}{\partial\eta_a} - ieA_a(\eta)\right)\right] + 2 - imR \right\}\chi = 0 \, , \tag{132}$$

which is a simple generalization of Eq. (40). The electron propagator corresponding to Eq. (132) will of course differ from the massless propagator of Eq. (9), but the electron-photon vertex and the photon propagator will be unchanged. The massive 5-dimensional formalism is not exactly equivalent to ordinary massive electrodynamics in Minkowski space-time, but as Dirac[5] has shown, in any finite neighborhood of $\eta_5 = R$, Eq. (132) reduces to the usual x-space Dirac equation in the limit $R \to \infty$. It is only in the completely massless case that the 5-dimensional and x-space formalisms have the same physical content.

It should be emphasized that while our electron wave equation is identical (in the case $m = 0$) to Dirac's, our treatment of the Maxwell equations is substantially different. Unlike our expressions for the electromagnetic field strengths, which involve $\partial/\partial\eta_a$ only through the angular momentum operator L_{ab}, Dirac's expressions[5] involve $\partial/\partial\eta_a$ by itself. Hence, in order to avoid going off the hypersurface of constant η^2, Dirac finds it necessary to introduce homogeneity constraints on the electromagnetic potential, of the type encountered in the O(5,1)-covariant formalism.

In our formulation of the 5-dimensional theory, such constraints are unnecessary, and an examination of the 5-dimensional Feynman rules of Eq. (9) shows, in fact, that they are not homogeneous in the coordinates. The absence of homogeneity requirements permits eigenfunction expansions of the field operators, and should therefore make possible a canonical quantization of the 5-dimensional formalism.[15] The first step in canonical quantization would be to write down an appropriate Lagrangian density; it is readily seen that variation of

$$\mathcal{L} = -\frac{1}{12}(F_{abc})^2 + \bar{\chi}\left\{i\gamma_{ab}\left[\eta_a\left(\frac{\partial}{\partial\eta_b} - ieA_b(\eta)\right) - \eta_b\left(\frac{\partial}{\partial\eta_a} - ieA_a(\eta)\right)\right] + 2 - imR\right\}\chi \tag{133}$$

gives the correct equations of motion. The next step is to pick a "time" axis for purposes of ordering operator products in Green's functions and for the definition of canonical momenta. Ordering with respect to $\eta_4/(1+\eta_5)$ would correspond to the conventional time-ordering in x-space. Another possible choice, ordering with respect to η_5, would correspond to x-space ordering with respect to x^2, a form of quantization which has been recently studied by Fubini and Jackiw.[16]

This concludes our discussion of possible avenues for generalization of our results.[17] Let us next briefly consider possible calculational advantages of the 5-dimensional formalism for massless electrodynamics. The key point to notice is that whereas the wave operator in Euclidean x-space is $\Box_x^2$, with a continuum spectrum $-p^2 = -(\text{momentum})^2$, the wave operator on the hypersphere is $L^2 - 4$, with discrete spectrum $-2(n+1)(n+2)$. This difference in spectra has two important consequences. First, the fact that the spectrum of $\Box_x^2$ contains 0 leads to the occurrence of infrared divergences in x-space calculations of propagators and vertex parts. These divergences are known to cancel, however, in closed fermion vacuum polarization loops,[18] and this is reflected in our ability to map vacuum polarization calculations onto the unit hypersphere, where the spectrum of the wave operator does not contain 0. In other words, closed fermion loop calculations on the unit hypersphere are manifestly infrared-finite. Second, it is difficult to see how to introduce approximations in massless electrodynamics when calculating in Euclidean x-space, since there is no natural scale for selecting one region of p^2 as being more important than another. [We have particularly in mind the calculation of the function $F^{[1]}(\alpha)$ defined in Sec. 1, where no natural scale for making approximations is provided by external momenta.] The situation is different on the hypersphere, where unity is a natural scale for measuring the spectrum $-2(n+1)(n+2)$, and where the semiclassical region of large quantum numbers, $n >> 1$, provides a natural domain for making approximations. The development of techniques for making such semiclassical approximations on the hypersphere is an important problem, which, hopefully, may shed light on the nature of the elusive function $F^{[1]}(\alpha)$.

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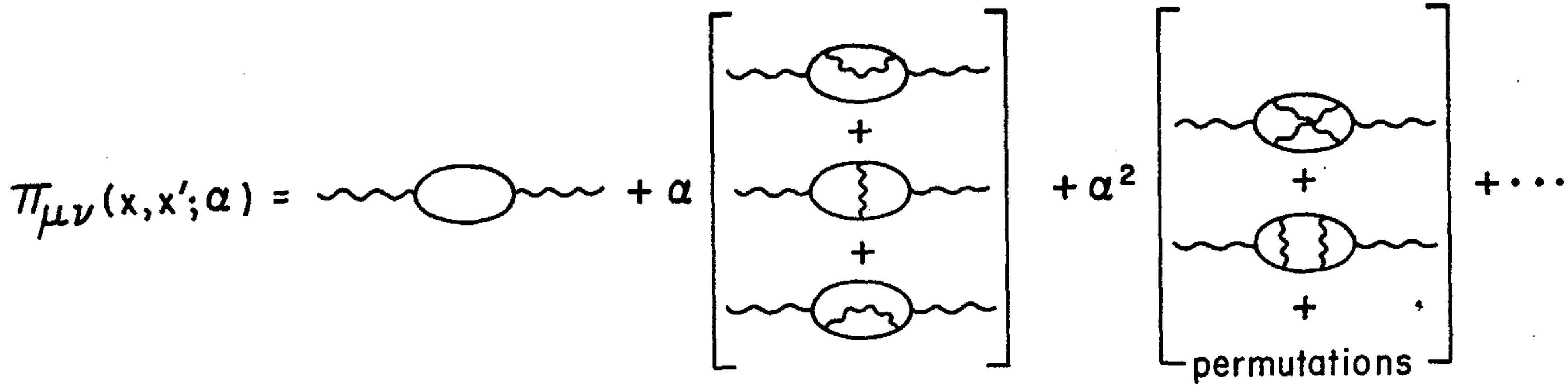
FIGURE CAPTIONS

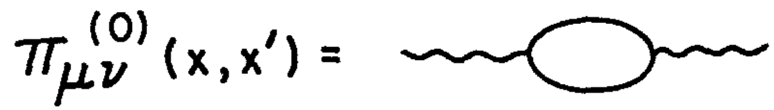
Fig. 1: (a) Single-fermion-loop vacuum polarization diagrams in spin-$\frac{1}{2}$ electrodynamics.

(b) Lowest order vacuum polarization diagram.



$\pi_{\mu\nu}(x,x';\alpha) =$
$+\ \alpha$
+
+
$+\ \alpha^2$
+
+
permutations
$+ \cdots$

(a)



$\pi^{(0)}_{\mu\nu}(x,x') =$

(b)