



LIGHT CONE APPROACH TO STRUCTURE FUNCTION INEQUALITIES

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ABSTRACT

Scaling limit inequalities on the structure functions for inelastic lepton-nucleon scattering are considered in the light cone model of Fritzsch and Gell-Mann. The connection with Nachtmann's parton model approach is clarified.

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The quark-parton model implies certain interesting positivity restrictions on the structure functions of inelastic lepton-hadron scattering in the scaling limit, as has been shown by Nachtmann.<sup>1</sup> It is by now common lore that the general features of the model are summarized in an alternative approach based on the Fritzsch, Gell-Mann<sup>2</sup> conjecture concerning the leading light cone singularities of current commutators. On this scheme, or its simple variants,<sup>3</sup> the currents are built up in the usual way out of quark fields, and the light cone commutator is assumed to have the same tensor and SU(3) structure that one would compute for a free field theory. In the context of his positivity conditions Nachtmann already noted briefly the identity of the two approaches, basing himself on unpublished work of one of the present authors (C.G.C.). The others of us, separately and in pairs, had likewise worked out the positivity conditions directly from the light cone approach. It seems to us worthwhile now to present the main steps of this light cone approach. At an early stage in the arithmetic a certain technical transformation quickly reveals the connection with the parton model.

We consider a nonet of currents formed out of quark fields according to

$$j_{\mu}^a = i\bar{q}\gamma_{\mu}(1 + \gamma_5)\frac{\lambda^a}{2}q, \quad (1)$$

where  $a = 0, 1, 2, \dots, 8$  is a  $U(3)$  label; the  $\lambda^a$ , for  $a \neq 0$ , are the usual  $SU(3)$  matrices, and  $\lambda^0$  is proportional to the unit matrix, with proportionality constant  $(2/3)^{1/2}$ . Then uniformly

$$\text{Tr } \lambda^a \lambda^b = 2 \delta_{ab}, \quad (2)$$

and

$$\lambda^a \lambda^b = (if^{abc} + d^{abc}) \lambda^c, \quad (3)$$

with

$$f^{abo} = 0, \quad d^{abo} = (2/3)^{1/2} \delta_{ab}, \quad a, b = 0, 1, \dots, 8. \quad (4)$$

Now consider a multiplet of hadrons,  $\alpha, \beta, \dots$ , belonging to an irreducible representation of some presumed strong interaction symmetry, e.g.,  $SU(2)$  or  $SU(3)$ ; and consider the forward current-hadron scattering process

$$b + \beta \rightarrow a + \alpha,$$

where the Greek letters specify the hadrons; Latin letters, the currents.

The absorptive amplitude is characterized by a familiar set of structure functions. We restrict ourselves to the Bjorken scaling limit,  $\omega = q^2/2m\nu$  fixed,  $q^2 \rightarrow \infty$ ; and we denote the structure function by  $(F_i)_{\alpha\beta}^{ab}$ . On the Fritzsch, Gell-Mann model for the structure of light cone commutators one finds

$$2\omega (F_1)_{\alpha\beta}^{ab} = (F_2)_{\alpha\beta}^{ab}$$

and

$$(F_{\pm})_{\alpha\beta}^{ab} \equiv (F_1)_{\alpha\beta}^{ab} \mp 1/2 (F_3)_{\alpha\beta}^{ab} = (\pm if^{abc} + d^{abc})(G_{\pm}^c)_{\alpha\beta}, \quad (5)$$

where the hadron matrix elements  $[G_{\pm}^c(\omega)]_{\alpha\beta}$  are  $\omega$ -dependent functions determined by detailed dynamics which are not specified by the model. The structure functions  $F_1^{(\nu)}$  and  $F_3^{(\nu)}$  for  $\Delta S = 0$  neutrino processes are given by

$$F_1^{(\nu)} = -(G_+^3 - G_-^3) + \frac{1}{\sqrt{3}}(G_+^8 + G_-^8) + \sqrt{2/3}(G_+^0 + G_-^0) \quad (6)$$

$$F_3^{(\nu)} = 2(G_+^3 + G_-^3) - \frac{2}{\sqrt{3}}(G_+^8 - G_-^8) - 2\sqrt{2/3}(G_+^0 - G_-^0), \quad (7)$$

where we have suppressed the indices  $\alpha$  and  $\beta$ , so that the above objects can be considered as operators in the space of hadron states. The structure function  $F_1^{(\gamma)}$  for scattering of the electromagnetic current  $[i\bar{q}\gamma_{\mu}Qq]$  is given by

$$F_1^{(\gamma)} = \frac{1}{6} (G_+^3 + G_-^3) + \frac{1}{6\sqrt{3}} (G_+^8 + G_-^8) + \frac{1}{3}\sqrt{2/3} (G_+^0 + G_-^0). \quad (8)$$

Following Nachtmann we now observe, from the positivity of cross sections, that

$$C_\alpha^{a*} (F_\pm)^{ab} C_\beta^b \geq 0, \quad (9)$$

for arbitrary choices of the coefficients  $C_Y^c$ . This implies that the matrices  $F_+$  and  $F_-$  must have non-negative eigenvalues. These matrices depend on the matrix elements  $(G_\pm^c)_{\alpha\beta}$ . The latter, in turn, are expressible in terms of certain reduced matrix elements according to a pattern determined by the strong interaction symmetries. In this connection one can reliably invoke SU(2) symmetry; but we shall also want to consider the less exact case of SU(3) symmetry. At the end we are only interested here in the electromagnetic and weak  $\Delta S = 0$  structure functions for protons and neutrons, hence only in the diagonal nucleon matrix elements of  $G_\pm^c$ ,  $c = 0, 3, 8$ . Nevertheless, in order to extract the most stringent positivity conditions on these objects we are entitled to consider the full nonet of currents and, for the SU(3) case, the full octet of baryons.

In the case where we only invoke SU(2) symmetry, it is in fact enough to restrict attention to nucleons and to currents  $J_\mu^a$  with zero

hypercharge,  $a = 0, 1, 2, 3,$  and  $8$ . Then  $F_+$  (and similarly  $F_-$ ) is a  $10 \times 10$  matrix carrying the  $SU(2)$  representations  $3/2, 1/2, 1/2, 1/2$ . It is parameterized by three reduced matrix elements which characterize the nucleon matrix elements of  $G_+^a$ ,  $a = 0, 1, 2, 3, 8$ . Owing to  $SU(2)$  invariance for the strong interactions,  $F_+$  can at most have four distinct eigenvalues and we might therefore anticipate four positivity conditions on the three reduced matrix elements. When  $SU(3)$  symmetry is invoked for the strong interactions, we are led to consider the full nonet of currents and full octet of baryons. Here  $F_+$  (and similarly  $F_-$ ) is a  $72 \times 72$  matrix carrying the  $SU(3)$  representations  $27, 10, \overline{10}, 8, 8, 8, 1$ . It is again parameterized by three reduced matrix elements which specify the  $(G_+^c)_{\gamma\beta}$ . Insofar as  $SU(3)$  is a valid strong interaction symmetry  $F_+$  can at most have seven distinct eigenvalues; and we might therefore anticipate seven positivity conditions on the three reduced matrix elements.

As so far described the strategy is based solely on the light cone results of Eq. (5) and on the positivity condition of Eq. (9). One constructs the matrices  $F_{\pm}$  and imposes positivity conditions on the eigenvalues. In actual fact matters are less complicated than the previous discussion has suggested. On the basis of an elementary observation we can simplify the arithmetic and at the same time make contact with the quark parton model. Let us consider the matrix  $F_+$  (similar considerations will hold for  $F_-$ ). From Eqs. (2) and (3) we observe that

$$if^{abc} + d^{abc} = \frac{1}{2} \text{Tr} \lambda^a \lambda^b \lambda^c, \quad (10)$$

hence that

$$(F_+)^{a,b}_{\alpha,\beta} = \text{Tr} [\lambda^a \lambda^b (\phi_+)_{\alpha\beta}], \quad (11)$$

where

$$\phi_+ = 1/2 \lambda^c G_+^c. \quad (12)$$

Notice that the  $(\phi_+)_{\alpha\beta}$  are 3 x 3 matrices. Indeed, we may regard the matrix elements  $(\phi_+)_{\alpha\beta}^{ji}$  as describing the scattering of antiquarks on hadrons:

$$i + \beta \rightarrow j + \alpha,$$

where the indices i and j specify the antiquarks. For the matrix  $F_-$  we proceed in the same way, writing

$$(F_-)^{ab}_{\alpha\beta} = \text{Tr} [\lambda^b \lambda^a (\phi_-)_{\alpha\beta}] \quad (13)$$

$$\phi_- = 1/2 \lambda^c G_-^c \quad (14)$$

The matrix elements  $(\phi_-)_{\alpha\beta}^{ji}$  describe the scattering of quarks on hadrons.

In the matrices  $\phi_{\pm}$  one now recognizes the parton model description

discussed by Nachtmann, and from here on the treatments are identical. It is easy to see from Eq. (9) that the matrices  $\phi_{\pm}$  must have non-negative eigenvalues; i. e., for positivity of the eigenvalues of  $F_{\pm}$  it is necessary and sufficient that the  $\phi_{\pm}$  have positive eigenvalues. Arithmetically it is of course easier to diagonalize the  $\phi_{\pm}$  matrices, but this is a technicality. The important thing is the connection with the parton model. To make matters more explicit, we shall briefly repeat the remaining steps, already carried out by Nachtmann in the quark parton model context.

(i) SU(2): -

The reduced matrix elements here (in a bra, ket notation) may be taken to be

$$\langle p | G_{\pm}^3 | p \rangle = - \langle n | G_{\pm}^3 | n \rangle \equiv a_{\pm}, \quad (15.1)$$

$$\langle p | G_{\pm}^8 | p \rangle = \langle n | G_{\pm}^8 | n \rangle \equiv \sqrt{3} b_{\pm} \quad (15.2)$$

$$\langle p | G_{\pm}^0 | p \rangle = \langle n | G_{\pm}^0 | n \rangle \equiv \sqrt{3/2} c_{\pm}. \quad (15.3)$$

The physically interesting structure functions are parametrized by these objects, as are the matrices  $\phi_{\pm}$ . From the requirement that the latter have non negative eigenvalues, we learn after some arithmetic that

$$\bar{+} a_{\pm} + b_{\pm} + c_{\pm} \equiv u_{\pm} \geq 0 \quad (16.1)$$

$$\pm 3 a_{\pm} + b_{\pm} + c_{\pm} \equiv v_{\pm} \geq 0 \quad (16.2)$$

$$-2 b_{\pm} + c_{\pm} \equiv w_{\pm} \geq 0 \quad (16.3)$$

Expressing the structure functions in terms of these non-negative quantities, we find

$$F_1^{(\gamma p)} = 1/36 \left\{ 6u_+ + 4v_+ + 2w_+ + 9u_- + v_- + 2w_- \right\} \quad (17.1)$$

$$F_1^{(\gamma n)} = 1/36 \left\{ 9u_+ + v_+ + 2w_+ + 6u_- + 4v_- + 2w_- \right\} \quad (17.2)$$

$$F_1^{(\nu p)} = \left\{ u_+ + u_- \right\} \quad (17.3)$$

$$F_1^{(\nu n)} = 1/2 \left\{ u_+ + v_+ + u_- + v_- \right\} \quad (17.4)$$

$$F_3^{(\nu p)} = 2 \left\{ -u_+ + u_- \right\} \quad (17.5)$$

$$F_3^{(\nu n)} = - \left\{ u_+ + v_+ - u_- - v_- \right\} \quad (17.6)$$

Analogous expressions are given by Nachtmann, in a parton model

context where the  $u_{\pm}$ ,  $v_{\pm}$ ,  $w_{\pm}$  are related to triplets of antiquarks and quark probability densities. We do not bother here to spell out the translation table.

(ii) SU(3): -

Here we deal with the full nonet of currents and octet of baryons, introducing the reduced matrix elements  $A_{\pm}$ ,  $B_{\pm}$ ,  $C_{\pm}$  according to

$$(G_{\pm}^c)_{\alpha\beta} = i f_{\alpha c\beta} A_{\pm} + d_{\alpha c\beta} B_{\pm} + \frac{1}{\sqrt{6}} (C_{\pm} - 2B_{\pm}) \delta_{\alpha\beta} \delta_{co}. \quad (18)$$

In particular

$$\langle p | G_{\pm}^3 | p \rangle = - \langle n | G_{\pm}^3 | n \rangle = A_{\pm} + B_{\pm}. \quad (19.1)$$

$$\langle p | G_{\pm}^8 | p \rangle = \langle n | G_{\pm}^8 | n \rangle = \sqrt{3} (A_{\pm} - \frac{1}{3} B_{\pm}). \quad (19.2)$$

$$\langle p | G_{\pm}^0 | p \rangle = \langle n | G_{\pm}^0 | n \rangle = \sqrt{2/3} C_{\pm}. \quad (19.3)$$

From the requirement that the  $\phi_{\pm}$  have non negative eigenvalues, we learn that

$$\mp 3 A_{\pm} + B_{\pm} + C_{\pm} \equiv 3U_{\pm} \geq 0 \quad (20.1)$$

$$\pm 3 A_{\pm} - 5B_{\pm} + C_{\pm} \equiv 3V_{\pm} \geq 0 \quad (20.2)$$

$$\pm 9 A_{\pm} + 5B_{\pm} + C_{\pm} \equiv 3W_{\pm} \geq 0 \quad (20.3)$$

where the factors 3 on the right hand sides are for convenience.

Then

$$F_1^{(\gamma p)} = 1/24 \left\{ 8 U_+ + 4V_+ + 4W_+ + 13U_- + 2V_- + W_- \right\} \quad (21.1)$$

$$F_1^{(\gamma n)} = 1/24 \left\{ 9U_+ + 6V_+ + W_+ + 9U_- + 6V_- + W_- \right\} \quad (21.2)$$

$$F_1^{(\nu p)} = \left\{ U_+ + V_+ + 2U_- \right\} \quad (21.3)$$

$$F_1^{(\nu n)} = 1/4 \left\{ 3U_+ + 2V_+ + 3W_+ + 4U_- + 4V_- \right\} \quad (21.4)$$

$$F_3^{(\nu p)} = 2 \left\{ -U_+ - V_+ + 2U_- \right\} \quad (21.5)$$

$$F_3^{(\nu n)} = 1/2 \left\{ -3U_+ - 2V_+ - 3W_+ + 4U_- + 4V_- \right\} \quad (21.6)$$

For the case of SU(2) symmetry, Eqs. (17) imply one equality among the structure functions:<sup>4</sup>

$$12 (F_1^{(\gamma p)} - F_1^{(\gamma n)}) = F_3^{(\nu p)} - F_3^{(\nu n)}$$

In addition, various inequalities follow from the positivity of the quantities  $u_{\pm}$ ,  $v_{\pm}$ ,  $w_{\pm}$ . The stronger assumption of SU(3) symmetry does not produce any new equalities. The inequality relations, however, are in some cases more stringent than those following from SU(2). Thus, as noted by Nachtmann, the SU(2) inequalities

$$\text{SU(2):} \quad 4 \geq F_1^{(\gamma n)} / F_1^{(\gamma p)} \geq 1/4 \quad (22)$$

get replaced for SU(3) by

$$\text{SU(3):} \quad 3 \geq F_1^{(\gamma n)} / F_1^{(\gamma p)} \geq 1/4. \quad (23)$$

It may also be of some interest to note the inequalities

$$\frac{F_1^{(\nu p)}}{4F_1^{(\nu n)} - F_1^{(\nu p)}} \leq \begin{cases} 12/5, & \text{SU(2)} \\ 48/23, & \text{SU(3)} \end{cases} \quad (24)$$

We call attention to this because of the experimental indication that  $F_1^{(\nu n)}/F_1^{(\nu p)}$  may be pressing close to the lower limit of Eqs. (22) or (23), as  $\omega \rightarrow 1$ . If the inequality is in fact violated, the quark model (in the parton or light cone version) would of course have to be abandoned. If the inequality is satisfied, but with  $F_1^{(\nu n)}/F_1^{(\nu p)}$  very close to its lower limit in some range of  $\omega$ , then Eqs. (24) serve to set a stringent upper bound on  $F_1^{(\nu p)} = F_2^{(\nu p)}/2\omega$  in the corresponding range of  $\omega$ .

The structure function inequalities discussed here may become stronger under more specific dynamical conditions. For example, if the scaling regime becomes diffractive as  $\omega \rightarrow 0$ , one finds for  $\omega \rightarrow 0$

$$F_1^{(\nu p)} \leq F_1^{(\nu p)} \begin{cases} 18/5, & \text{SU(2)} \\ 288/85 & \text{SU(3)} \end{cases} \quad (25)$$

In general, the Eqs. (17) or (21) can be used by the interested reader to form his own inequalities

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